

# Combinatorial Modulus on Boundary of Right-Angled Hyperbolic Buildings

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## Abstract

In this article, we discuss the quasiconformal structure of boundaries of right-angled hyperbolic buildings using combinatorial tools. In particular we exhibit some examples of buildings of dimension 3 and 4 whose boundaries satisfy the combinatorial Loewner property. This property is a weak version of the Loewner property. This is motivated by the fact that the quasiconformal structure of the boundary led to many results of rigidity in hyperbolic spaces since G.D. Mostow. In the case of buildings of dimension 2, many work have been done by M. Bourdon and H. Pajot. In particular, the Loewner property on the boundary permitted them to prove the quasi-isometry rigidity of right-angled Fuchsian buildings.

**Keywords:** Boundary of hyperbolic space, building, combinatorial modulus, combinatorial Loewner property, quasi-conformal analysis.

## 1 Introduction

### 1.1 Starting point

The origin of the theory of modulus of curves in compact metric spaces must be found in the classical theory of quasiconformal maps in Euclidean spaces (see [Väi71] or [Vuo88]). Quasiconformal maps are maps between homeomorphisms and bi-Lipschitz maps. The aim of the classical theory is to describe the regularity of quasiconformal maps in  $\mathbb{R}^d$  and to exhibit invariants under these maps. The notion of abstract Loewner space, introduced by J. Heinonen and P. Koskela (see [HK98] or [Hei01]), intends to describe metric measured spaces whose quasiconformal maps have a behavior of Euclidean flavor.

Moreover, since G.D. Mostow it is known that the quasiconformal structure of the boundary of a hyperbolic space controls the geometry of the space. It turns out that this idea extends to the case of Gromov hyperbolic spaces and groups. Finding a Loewner space as the visual boundary of a Gromov hyperbolic group has been useful to establish rigidity results about the group (see [Hai09b] for a survey on those results). The idea that one wants to use to prove rigidity is that the quasi-isometries of an hyperbolic space are given by the quasimetric homeomorphisms of the boundary. The Loewner property makes it possible because the classes of quasi-Möbius, quasimetric and quasiconformal maps are equal in a Loewner space.

It is difficult to prove that the boundary of a hyperbolic space is a Loewner space. To do so, one needs to find a measure on the boundary that is optimal for the *conformal dimension*. This quasiconformal invariant has been introduced by P. Pansu in [Pan89]. Finding a measure that realizes the conformal dimension and even computing the conformal dimension, are very difficult questions that we can solve in few examples for the moment.

An interesting example of this kind is the work done by M. Bourdon and H. Pajot in Fuchsian buildings. They proved that the boundary of these buildings are Loewner spaces and then used this structure to prove the quasi-isometry rigidity of these buildings (see [BP00]).

Buildings are singular spaces introduced by J. Tits to study exceptional Lie groups. Currently buildings became a topic of interest by themselves. Among them, right-angled buildings have been classified by F. Haglund and F. Paulin in [HP03]. They are equipped with a wall structure and with a simply transitive group action on the chambers that make them very regular objects. Fuchsian buildings are right-angled hyperbolic buildings of dimension 2. In light of the results by M. Bourdon and H. Pajot in dimension 2, we have the questions:

**Question 1.1.** *Are higher dimensional right-angled hyperbolic buildings rigid? What are the quasiconformal properties of their boundaries?*

The geometry of higher dimensional right-angled buildings is very close to the geometry of Fuchsian buildings. This gives hope that these questions may have interesting answers. However, the methods used in Fuchsian buildings are very specific to the dimension 2. Thus these questions are not easy. In this article, we will use *combinatorial modulus* for a first approach of the conformal structure of higher dimensional right-angled buildings.

A major rigidity question related to the quasiconformal structure on the boundary is the following conjecture due to J.W. Cannon.

**Conjecture 1.2** ([CS98, Conjecture 5.1.]). *If  $\Gamma$  is a hyperbolic group and  $\partial\Gamma$  is homeomorphic to  $\mathbb{S}^2$ , then  $\Gamma$  acts geometrically on  $\mathbb{H}^3$ .*

In particular, this conjecture implies Thurston's hyperbolization conjecture of 3-manifolds. Although Thurston's conjecture is now a theorem by G. Perelman, Cannon's conjecture remains very interesting as it is logically independent of Thurston's conjecture.

The combinatorial modulus have been introduced by J.W. Cannon in [Can94] and by M. Bonk and B. Kleiner in [BK02] during the investigation of the quasiconformal structure of the 2-spheres to approach the conjecture and by P. Pansu in a more general context in [Pan89]. Combinatorial modulus gave birth to a weak version of the Loewner property: the *Combinatorial Loewner Property* (CLP). One of the feature of these modulus is that they can be used to characterize the conformal dimension as a critical exponent on the boundary.

Recently M. Bourdon and B. Kleiner (see [BK13]) gave examples of boundaries of Coxeter groups that satisfy the CLP but that are not known for satisfying the Loewner property. They used this property to give a new proof of Cannon's conjecture for Coxeter groups. Some of the methods they used for Coxeter groups can be adapted to the case of right-angled buildings. This was a motivation to investigate higher dimensional right-angled buildings using combinatorial modulus.

## 1.2 Main result

In this article, we use the combinatorial modulus to investigate the quasiconformal boundary of right-angled hyperbolic buildings. Thanks to methods in [BK13], we obtain a control of the combinatorial modulus on the boundary in terms of the curves contained in *parabolic limit sets* (see Section 6). Then we introduce a *weighted modulus* on the boundary of the apartments. This allows us to control the modulus in the building by a modulus in the apartment (see Section 8). For well chosen examples, the boundary of the apartment has a lot of symmetries that provide a strong control of the modulus. In particular, we exhibit some examples of hyperbolic buildings in dimension 3 and 4 whose boundaries satisfy the CLP.

**Theorem 1.3** (Corollary 10.3). *Let  $D$  be the right-angled dodecahedron in  $\mathbb{H}^3$  or the right-angled 120-cell in  $\mathbb{H}^4$ . Let  $W_D$  be the hyperbolic reflection group generated by reflections about the faces of  $D$ . Let  $\Delta$  be the right-angled building of constant thickness  $q \geq 3$  and of Coxeter group  $W_D$ . Then  $\partial\Delta$  satisfies the CLP.*

Along with this result we also give in Theorem 9.1, a characterization of the conformal dimension of the building using a critical exponent computed in an apartment.

## 1.3 Organization of the article

In Section 2, we introduce the combinatorial modulus of curves in the general setting of compact metric spaces. Then in Section 3, we restrict to the case of boundaries of hyperbolic spaces.

After these reminders, we give the main steps and ideas of the proof of Theorem 10.1 in Section 4. This section is essentially a summary of the article.

Then, in Section 5, we describe the geometry of locally finite right-angled hyperbolic buildings.

The key notion of parabolic limit sets is introduced in Section 6 where we study the modulus of curves in parabolic limit sets. This section is based on ideas used in Coxeter groups in [BK13, Sections 5 and 6]. In particular, Theorem 6.12 is the first major step towards the proof of Theorem 10.1. As a consequence of this theorem, we obtain a first application to the CLP (Theorem 6.13).

In Section 7, we describe the combinatorial metric on the boundary of the group in terms of the geometry of the building. This metric is useful in computing the combinatorial modulus. Then, in Section 8 we discuss how the modulus in the boundary of an apartment may be related to a modulus in the boundary of the building. In particular, Theorem 8.9 is the second major step necessary to prove Theorem 10.1. We use this theorem to prove Theorem 9.1 which relates the conformal dimension of the boundary of the building to a critical exponent computed in the boundary of an apartment.

In Section 9, we add the constant thickness assumption for the building under which the results of the preceding section can be made more precise. In particular, we find that the conformal dimension of the boundary of the building is equal to a critical exponent computed in the boundary of an apartment (see Theorem 9.1). Finally in Section 10, we gather these tools to obtain examples of right-angled-buildings of dimension 3 and 4 whose boundary satisfies the CLP (see Corollary 10.3).

## 1.4 Terminology and notation

Throughout this paper, we will use the following conventions. The identity element in a group will always be designated by  $e$ . For a set  $E$ , the *cardinality* of  $E$  is designated by  $\#E$ . A *proper* subset  $F$  of  $E$  is a subset  $F \subsetneq E$ .

If  $\mathcal{G}$  is a graph then  $\mathcal{G}^{(0)}$  is the *set of vertices* of  $\mathcal{G}$  and  $\mathcal{G}^{(1)}$  is the *set of edges* of  $\mathcal{G}$ . For  $v, w \in \mathcal{G}^{(0)}$ , we write  $v \sim w$  if there exists an edge in  $\mathcal{G}$  whose extremities are  $v$  and  $w$ . If  $V \subset \mathcal{G}^{(0)}$ , the *full subgraph* generated by  $V$  is the graph  $\mathcal{G}_V$  such that  $\mathcal{G}_V^{(0)} = V$  and an edge lies between two vertices  $v, w$  if and only if there exists an edge between  $v$  and  $w$  in  $\mathcal{G}$ . A full subgraph is called a *circuit* if it is a cyclic graph  $C_n$  for  $n \geq 3$ . A graph is called a *complete graph* if for any pair of distinct vertices  $v, w$  there exists an edge between  $v$  and  $w$ .

A *curve* in a compact metric space  $(Z, d)$  is a continuous map  $\eta : [0, 1] \rightarrow Z$ . Usually, we identify a curve with its image. If  $\eta$  is a curve in  $Z$ , then  $\mathcal{U}_\epsilon(\eta)$  denotes the  $\epsilon$ -neighborhood of  $\eta$  for the  $C^0$ -topology. This means that a curve  $\eta' \in \mathcal{U}_\epsilon(\eta)$  if and only if there exists  $s : t \in [0, 1] \rightarrow [0, 1]$  a parametrization of  $\eta$  such that for any  $t \in [0, 1]$  one has  $d(\eta(s(t)), \eta'(t)) < \epsilon$ .

In a metric space  $Z$ , if  $A \subset Z$  then  $N_r(A)$  is the  $r$ -neighborhood of  $A$ . The *closure* of  $A$  is designated by  $\overline{A}$  and the *interior* of  $A$  by  $\text{Int}(A)$ . If  $B = B(x, R)$  is an open ball and  $\lambda \in \mathbb{R}$  then  $\lambda B$  is the ball of radius  $\lambda R$  and of center  $x$ . A ball of radius  $R$  is called an  $R$ -ball. The closed ball of center  $x$  and radius  $R$  is designated by  $\overline{B}(x, R)$ .

A *geodesic line* (resp. *ray*) in a metric space  $(Z, d)$  is an isometry from  $(\mathbb{R}, |\cdot - \cdot|)$

(resp.  $([0, +\infty), |\cdot - \cdot|)$ ) into  $(Z, d)$ . The real hyperbolic space (resp. Euclidean space) of dimension  $d$  is denoted  $\mathbb{H}^d$  (resp.  $\mathbb{E}^d$ ).

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## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Starting point . . . . .	1
1.2	Main result . . . . .	3
1.3	Organization of the article . . . . .	3
1.4	Terminology and notation . . . . .	4
1.5	Acknowledgement . . . . .	5
<b>2</b>	<b>Combinatorial modulus and the CLP</b>	<b>6</b>
2.1	General properties of combinatorial modulus of curves . . . . .	7
2.2	Combinatorial Loewner property . . . . .	9
2.3	Loewner spaces . . . . .	10
2.4	First properties and examples . . . . .	12
<b>3</b>	<b>Combinatorial modulus of curve families on boundaries of hyperbolic groups</b>	<b>15</b>
3.1	Boundaries of hyperbolic groups and approximately self-similar spaces . . .	15
3.2	Combinatorial modulus and conformal dimension . . . . .	17
3.3	How to prove the CLP . . . . .	19
<b>4</b>	<b>Steps in the proof of Theorem 10.1</b>	<b>20</b>
<b>5</b>	<b>Locally finite right-angled hyperbolic buildings</b>	<b>22</b>
5.1	Chamber systems . . . . .	22
5.2	Coxeter systems . . . . .	23
5.3	The Davis chamber of $(W, S)$ . . . . .	24
5.4	Buildings . . . . .	25
5.5	Graph products and right-angled buildings . . . . .	26

5.6	The Davis complex associated with $\Gamma$ . . . . .	27
5.7	Building-walls and residues in the Davis complex . . . . .	27
5.8	Geometric characterization of parabolic subgroups . . . . .	30
5.9	$\Sigma$ as a metric space . . . . .	31
5.10	Boundary of the building . . . . .	32
<b>6</b>	<b>Curves in connected parabolic limit sets</b>	<b>33</b>
6.1	Parabolic limit sets in $\partial\Gamma$ . . . . .	33
6.2	Modulus of curves in connected parabolic limit set . . . . .	36
6.3	Application to Fuchsian buildings . . . . .	41
<b>7</b>	<b>Combinatorial metric on boundaries of right-angled hyperbolic buildings</b>	<b>44</b>
7.1	Projections of chambers in $\Sigma$ . . . . .	44
7.2	Shadows on $\partial\Gamma$ . . . . .	47
7.3	Combinatorial metric on $\partial\Gamma$ . . . . .	51
7.4	Approximation of $\partial\Gamma$ with shadows . . . . .	54
<b>8</b>	<b>Modulus in the boundary of a building and in the boundary of an apartment</b>	<b>56</b>
8.1	Notations and conventions in $\partial A$ and in $\partial\Gamma$ . . . . .	56
8.2	Choice of approximations . . . . .	57
8.3	Weighted modulus in $\partial A$ . . . . .	58
8.4	Modulus in $\partial\Gamma$ compared with weighted modulus in $\partial A$ . . . . .	61
8.5	Consequences . . . . .	64
<b>9</b>	<b>Application to buildings of constant thickness</b>	<b>68</b>
<b>10</b>	<b>Dimension 3 and 4 right-angled buildings with boundary satisfying the CLP</b>	<b>70</b>

## 2 Combinatorial modulus and the CLP

The combinatorial modulus are tools that have been developed to compute modulus of curves in a metric space without a natural measure. The idea is to approximate the metric space with a sequence of finer and finer approximations. Then with these approximations we can construct discrete measures and compute combinatorial modulus. Finally, for well chosen examples we can check that this sequence of modulus has a good asymptotic behavior.

In this first section, we present the general theory of combinatorial modulus in compact metric spaces. We also recall basic definitions and facts about abstract Loewner spaces as

they inspired the theory of combinatorial modulus. Most of this section can be found in [BK13, Section 2] to which we refer for details.

In this section  $(Z, d)$  denotes a compact metric space.

## 2.1 General properties of combinatorial modulus of curves

For  $k \geq 0$  and  $\kappa > 1$ , a  $\kappa$ -approximation of  $Z$  on scale  $k$  is a finite covering  $G_k$  by open subsets such that for any  $v \in G_k$  there exists  $z_v \in v$  satisfying the following properties:

- $B(z_v, \kappa^{-1}2^{-k}) \subset v \subset B(z_v, \kappa 2^{-k})$ ,
- $\forall v, w \in G_k$  with  $v \neq w$  one has  $B(z_v, \kappa^{-1}2^{-k}) \cap B(z_w, \kappa^{-1}2^{-k}) = \emptyset$ .

A sequence  $\{G_k\}_{k \geq 0}$  is called a  $\kappa$ -approximation of  $Z$ .

**Example 2.1.** For  $k \geq 0$ , a  $2^{-k}$ -separated subset of  $Z$  is a subset  $E$  such that  $d(z, z') \geq 2^{-k}$  for any  $z \neq z' \in E_k$ . Since  $Z$  is compact any  $2^{-k}$ -separated subset of  $Z$  is finite. Let  $E_k$  be a  $2^{-k}$ -separated subset of  $Z$  of maximal cardinality. Then  $E_k$  satisfies the following property:

for any  $x \in Z$ , there exists  $z \in E_k$  such that  $d(x, z) \leq 2^{-k}$ .

The set  $\{B(z, 2^{-k})\}_{z \in E_k}$  defines a 2-approximation at scale  $k$  of  $Z$ .

Now we fix the approximation  $\{G_k\}_{k \geq 0}$ . We construct a discrete measure based on each  $G_k$  for  $k \geq 0$ . Let  $\rho : G_k \rightarrow [0, +\infty)$  be a positive function and  $\gamma$  be a curve in  $Z$ . The  $\rho$ -length of  $\gamma$  is

$$L_\rho(\gamma) = \sum_{\gamma \cap v \neq \emptyset} \rho(v).$$

For  $p \geq 1$ , the  $p$ -mass of  $\rho$  is

$$M_p(\rho) = \sum_{v \in G_k} \rho(v)^p.$$

Until the end of this subsection  $p \geq 1$  is fixed. Let  $\mathcal{F}$  be a non-empty set of curves in  $Z$ . We say that the function  $\rho$  is  $\mathcal{F}$ -admissible if  $L_\rho(\gamma) \geq 1$  for any curve  $\gamma \in \mathcal{F}$ .

**Definition 2.2.** The  $G_k$ -combinatorial  $p$ -modulus of  $\mathcal{F}$  is

$$\text{Mod}_p(\mathcal{F}, G_k) = \inf\{M_p(\rho)\}$$

where the infimum is taken over the set of  $\mathcal{F}$ -admissible functions and with the convention  $\text{Mod}_p(\emptyset, G_k) = 0$ .

The following equality is an alternative definition of the modulus:

$$\text{Mod}_p(\mathcal{F}, G_k) = \inf_{\rho} \frac{M_p(\rho)}{L_{\rho}(\mathcal{F})^p},$$

where the infimum is taken over the set of positive functions on  $G_k$  and with  $L_{\rho}(\mathcal{F}) = \inf_{\gamma \in \mathcal{F}} L_{\rho}(\gamma)$ .

The next proposition allows us to see the  $G_k$ -combinatorial  $p$ -modulus as a weak outer measure on the set of curves of  $Z$ . Usually, for an outer measure the subadditivity must hold over countable sets. This is useful to get intuition on these tools.

**Proposition 2.3** ([BK13, Proposition 2.1.]).

1. Let  $\mathcal{F}$  be a set of curves and  $\mathcal{F}' \subset \mathcal{F}$ . Then  $\text{Mod}_p(\mathcal{F}', G_k) \leq \text{Mod}_p(\mathcal{F}, G_k)$ .
2. Let  $\mathcal{F}_1, \dots, \mathcal{F}_n$  be families of curves. Then  $\text{Mod}_p(\bigcup_{i=1}^n \mathcal{F}_i, G_k) \leq \sum_{i=1}^n \text{Mod}_p(\mathcal{F}_i, G_k)$ .

A function  $\rho : G_k \rightarrow [0, +\infty)$  is called a *minimal function* for a set of curves  $\mathcal{F}$  if  $\text{Mod}_p(\mathcal{F}, G_k) = M_p(\rho)$ . Since we only compute finite sums, minimal functions always exist. Combining with a convexity argument, this also provides an elementary control of the modulus as follows. For  $\mathcal{F}$  a non-empty set of curves in  $Z$  and  $k \geq 0$

$$\frac{1}{(\#G_k)^{p-1}} \leq \text{Mod}_p(\mathcal{F}, G_k) \leq \#G_k.$$

In the rest of this article we mainly discuss the curves of  $Z$  of diameter larger than a fixed constant. For these curves the following basic property is useful.

**Proposition 2.4.** Let  $\mathcal{F}$  be a non-empty set of curves in  $Z$ . Assume that there exists  $d > 0$  such that  $\text{diam } \gamma \geq d$  for any  $\gamma \in \mathcal{F}$ . Then for any  $\epsilon > 0$ , there exists  $k_0 \geq 0$  such that for any  $k \geq k_0$ , there exists an admissible function  $\rho : G_k \rightarrow [0, +\infty)$  such that  $\rho(v) \leq \epsilon$  for any  $v \in G_k$ .

*Proof.* Let  $\gamma \in \mathcal{F}$ . We recall that  $\kappa$  denotes the multiplicative constant of the approximation  $\{G_k\}_{k \geq 0}$ . For  $k \geq \frac{\log(\kappa/d)}{\log 2}$ , as  $\text{diam } \gamma > d$  the following inequality holds

$$\#\{v \in G_k : v \cap \gamma \neq \emptyset\} \geq \frac{d}{\kappa 2^{-k}}.$$

Hence the constant function  $\rho : v \in G_k \rightarrow \frac{\kappa}{d} 2^{-k} \in [0, +\infty)$  is  $\mathcal{F}$ -admissible. This finishes the proof.  $\square$

A metric space  $Z$  is called *doubling* if there exists a uniform constant  $N$  such that each ball  $B$  of radius  $r$  is covered by  $N$  balls of radius  $r/2$ . In doubling spaces, the  $G_k$ -combinatorial  $p$ -modulus does not depend, up to a multiplicative constant, on the choice of the approximation.



**Proposition 2.5** ([BK13, Proposition 2.2.]). *Let  $(Z, d)$  be a compact doubling metric space. For each  $p \geq 1$ , if  $G_k$  and  $G'_k$  are respectively  $\kappa$  and  $\kappa'$ -approximations, there exists  $D = D(\kappa, \kappa')$  such that for any  $k \geq 0$*

$$D^{-1} \cdot \text{Mod}_p(\mathcal{F}, G_k) \leq \text{Mod}_p(\mathcal{F}, G'_k) \leq D \cdot \text{Mod}_p(\mathcal{F}, G_k).$$

Usually, we work with  $p \geq 1$  fixed and with approximately self-similar spaces (see Section 3). As these spaces are doubling, now we refer to the *combinatorial modulus on scale  $k$* , omitting  $p$  and the approximation.

## 2.2 Combinatorial Loewner property

In this subsection, we assume that  $(Z, d)$  is a compact arcwise connected doubling metric space. Let  $\kappa > 1$  and let  $\{G_k\}_{k \geq 0}$  denote a  $\kappa$ -approximation of  $Z$ . Moreover we fix  $p \geq 1$ .

A compact and connected subset  $A \subset Z$  is called a *continuum*. Moreover, if  $A$  contains more than one point,  $A$  is called a *non-degenerate* continuum. The relative distance between two disjoint non-degenerate continua  $A, B \subset Z$  is

$$\Delta(A, B) = \frac{\text{dist}(A, B)}{\min\{\text{diam } A, \text{diam } B\}}.$$

If  $A$  and  $B$  are two such continua,  $\mathcal{F}(A, B)$  denotes the set of curves in  $Z$  joining  $A$  and  $B$  and we write  $\text{Mod}_p(A, B, G_k) := \text{Mod}_p(\mathcal{F}(A, B), G_k)$ .

**Definition 2.6.** *Let  $p > 1$ . We say that  $Z$  satisfies the Combinatorial  $p$ -Loewner Property (CLP) if there exist two increasing functions  $\phi$  and  $\psi$  on  $(0, +\infty)$  with  $\lim_{t \rightarrow 0} \psi(t) = 0$ , such that*

- i) for any pair of disjoint non-degenerate continua  $A$  and  $B$  in  $Z$  and for all  $k \geq 0$  with  $2^{-k} \leq \min\{\text{diam } A, \text{diam } B\}$  one has:*

$$\phi(\Delta(A, B)^{-1}) \leq \text{Mod}_p(A, B, G_k),$$

- ii) for any pair of open balls  $B_1, B_2$  in  $Z$ , with same center and  $B_1 \subset B_2$ , and for all  $k \geq 0$  with  $2^{-k} \leq \text{diam } B_1$  one has:*

$$\text{Mod}_p(\overline{B_1}, Z \setminus B_2, G_k) \leq \psi(\Delta(\overline{B_1}, Z \setminus B_2)^{-1}).$$

As we assume that  $Z$  is doubling, thanks to Proposition 2.5, the CLP is independent of the choice of the approximation. As we noticed, the modulus on scale  $k$  is an outer measure (in a weak sense) over the set of curves in  $Z$ . With the previous remarks we can interpret intuitively the two inequalities of the definition as follows:

- i) there are plenty of curves joining two continua,*
- ii) the amount of curves joining two continua is a decreasing function of the relative distance.*

We present examples and properties about the CLP in Subsection 2.4.

### 2.3 Loewner spaces

Now we define the notion of Loewner space. This notion introduced in [HK98] has inspired the definition of the CLP. Moreover, the proof of many basic properties of combinatorial modulus are directly inspired by the classical theory of modulus (see [BK13]).

Now we consider  $(X, d, \mu)$  a metric measured space. For simplicity, we assume that  $X$  is compact and that  $(X, d, \mu)$  is a  $Q$ -Ahlfors-regular space ( $Q$ -AR or AR) for  $Q > 1$ . This means that there exists a constant  $C > 1$  such that for any  $0 < R \leq \text{diam } X$  and any  $R$ -ball  $B \subset X$  one has

$$C^{-1} \cdot R^Q \leq \mu(B) \leq C \cdot R^Q.$$

Note that under this assumption the measure  $\mu$  is comparable to the Hausdorff measure  $\mathcal{H}_d$ .

Let  $\mathcal{F}$  be a set of curves in  $X$ . A measurable function  $f : X \rightarrow [0, +\infty[$  is said to be  $\mathcal{F}$ -admissible if for any rectifiable curve  $\gamma \in \mathcal{F}$

$$\int_{\gamma(t)} f(\gamma(t)) dt \geq 1.$$

Note that the notion of admissibility does not use the measure on  $X$  but only the metric space structure.

**Definition 2.7.** *The  $Q$ -modulus of  $\mathcal{F}$  is*

$$\text{Mod}_Q(\mathcal{F}) = \inf \left\{ \int_X f^Q d\mu \right\}$$

where the infimum is taken over the set of  $\mathcal{F}$ -admissible functions and with the convention that  $\text{Mod}_Q(\mathcal{F}) = 0$  if  $\mathcal{F}$  does not contain rectifiable curves.

As before, if  $A$  and  $B$  are two disjoint non-degenerate continua,  $\mathcal{F}(A, B)$  denotes the set of curves in  $X$  joining  $A$  and  $B$ . Moreover, we write  $\text{Mod}_Q(A, B) := \text{Mod}_Q(\mathcal{F}(A, B))$ . In the literature on quasiconformal maps the pair  $(A, B)$  is called a *condenser* and the modulus (with respect to the Lebesgue measure)  $\text{Mod}_Q(A, B)$  the *capacity* of  $(A, B)$  (see [Vuo88]).

In  $X$ , the classical modulus are comparable to the combinatorial modulus in the following sense.

**Proposition 2.8** ([Hai09a, Prop B.2]). *Assume that  $X$  is equipped with an approximation  $\{G_k\}_{k \geq 0}$ . For  $d_0 > 0$ , let  $\mathcal{F}_0$  be the set of curves in  $X$  of diameter larger than  $d_0$ . Then for  $k$  large enough one has*

$$\text{Mod}_Q(\mathcal{F}_0, G_k) \asymp \text{Mod}_Q(\mathcal{F}_0),$$

if  $\text{Mod}_Q(\mathcal{F}_0) > 0$  and  $\lim \text{Mod}_Q(\mathcal{F}_0, G_k) = 0$  if  $\text{Mod}_Q(\mathcal{F}_0) = 0$ .

In addition for any pair  $A, B$  of non-degenerate disjoint continua and for  $k$  large enough one has

$$\text{Mod}_Q(A, B, G_k) \asymp \text{Mod}_Q(A, B)$$

if  $\text{Mod}_Q(A, B) > 0$  and  $\lim \text{Mod}_Q(A, B, G_k) = 0$  if  $\text{Mod}_Q(A, B) = 0$ .

Note that this connection between combinatorial and classical modulus is only valid for the dimension  $Q$ .

Now we can define Loewner spaces.

**Definition 2.9.** We say that  $(X, d, \mu)$  is a  $Q$ -Loewner space (or satisfies the  $Q$ -Loewner property) if there exists an increasing function  $\phi : (0, +\infty) \rightarrow (0, +\infty)$  such that for any pair of non-degenerate disjoint continua  $A$  and  $B$  in  $X$  one has:

$$\phi(\Delta(A, B)^{-1}) \leq \text{Mod}_Q(\mathcal{F}(A, B)).$$

We also say that  $X$  satisfies the *Loewner property* or the *classical Loewner property* to avoid the confusion with the CLP.

The control of the modulus from above is not required in this definition because it is automatically provided by the structure of  $Q$ -AR space.

**Theorem 2.10** ([HK98, Lemma 3.14.]). *There exists a constant  $C > 0$  such that the following property holds. Let  $A$  and  $B$  be two non-degenerate disjoint continua. Let  $0 < 2r < R$  and  $x \in X$  be such that  $A \subset \overline{B(x, r)}$  and  $B \subset X \setminus B(x, R)$ . Then*

$$\text{Mod}_Q(A, B) \leq C \left( \log \frac{R}{r} \right)^{1-Q}.$$

As a consequence there exists an increasing function  $\psi$  on  $(0, +\infty)$  with  $\lim_{t \rightarrow 0} \psi(t) = 0$ , such that for any pair of disjoint non-degenerate continua  $A$  and  $B$

$$\text{Mod}_Q(A, B) \leq \psi(\Delta^{-1}(A, B)).$$

More precisely, there exist some constants  $K, C > 0$  such that  $\psi(t) = K \left( \log(\frac{1}{t} + C) \right)^{1-Q}$  for any  $t > 0$ .

When  $X$  is a Loewner space, the asymptotic behavior of  $\phi$  is described in [HK98, Theorem 3.6.]. For  $t$  small enough  $\phi(t) \approx \log \frac{1}{t}$ , for  $t$  large enough  $\phi(t) \approx (\log t)^{1-Q}$ .

As we will see in the sequel an essential difference between the combinatorial and the classical modulus property is the importance of the dimension  $Q$  in the discussions about classical modulus.

## 2.4 First properties and examples

In this section  $Z$  is a compact arcwise connected doubling metric space and  $X$  is a compact  $Q$ -AR metric space ( $Q > 1$ ). First we recall a theorem and a conjecture that compare the CLP and the classical Loewner property.

**Theorem 2.11** ([BK13, Theorem 2.6.]). *If  $X$  is a compact  $Q$ -AR and Loewner metric space, then  $X$  satisfies the combinatorial  $Q$ -Loewner property.*

The next conjecture is a main motivation for studying group boundaries that satisfy the CLP.

**Conjecture 2.12** ([Kle06, Conjecture 7.5.]). *Assume that  $Z$  is quasi-Moebius homeomorphic to the boundary of a hyperbolic group. If  $Z$  satisfies the CLP then it is quasi-Moebius homeomorphic to a Loewner space.*

As announced we want to find and use the CLP on boundaries of hyperbolic groups. Quasi-isometries between hyperbolic spaces extend to quasi-Moebius homeomorphisms between the boundaries, so it is fundamental to know how the Loewner property and the CLP behave under quasi-Moebius maps. These maps have been introduced by J. Väisälä in [Väi85]. We recall that in a metric space  $(Z, d)$  the *cross-ratio* of four distinct points  $a, b, c, d \in Z$  is

$$[a : b : c : d] = \frac{d(a, b)}{d(a, c)} \cdot \frac{d(c, d)}{d(b, d)}.$$

For  $Z, Z'$  two metric spaces, an homeomorphism  $f : Z \rightarrow Z'$  is *quasi-Moebius* if there exists an homeomorphism  $\phi : [0, +\infty) \rightarrow [0, +\infty)$  such that for any quadruple of distinct points  $a, b, c, d \in Z$

$$[f(a) : f(b) : f(c) : f(d)] \leq \phi([a : b : c : d]).$$

If  $f$  is quasi-Moebius, as  $[a : c : b : d] = [a : b : c : d]^{-1}$ ,  $f^{-1} : Z' \rightarrow Z$  is also quasi-Moebius.

**Theorem 2.13** ([BK13, Theorem 2.6.]). *If  $Z'$  is quasi-Moebius homeomorphic to a compact space  $Z$  satisfying the CLP, then  $Z'$  also satisfies the CLP (with the same exponent).*

The Loewner property does not behave so well under quasi-Moebius maps. In particular, it is perturbed by a change of dimension whereas the CLP is not.

**Theorem 2.14** ([Tys98]). *Let  $X$  and  $X'$  be respectively  $Q$ -Loewner and  $Q'$ -AR compact metric spaces. Assume that  $X'$  is quasi-Moebius homeomorphic to  $X$ . Then  $X'$  is a Loewner space if and only if  $Q = Q'$ .*

If we apply to  $X$  a snowflake transformation  $f_\epsilon : (X, d) \rightarrow (X, d^\epsilon)$ ,  $0 < \epsilon < 1$  then  $\dim_{\mathcal{H}}(X, d^\epsilon) = \frac{1}{\epsilon} \dim_{\mathcal{H}}(X, d)$ . Such a transformation is a quasi-Moebius homeomorphism and Combining with the previous theorem we get the following fact.

**Fact 2.15.** *The Loewner property is not invariant under quasi-Möbius homeomorphisms.*

However quasi-Möbius maps are the appropriate homeomorphisms to discuss between Loewner spaces.

**Theorem 2.16** ([HK98]). *Let  $X$  and  $X'$  be compact  $Q$ -regular Loewner spaces and let  $f : X \rightarrow X'$  be a homeomorphism. The following are equivalent*

1.  *$f$  is quasi-Möbius,*
2. *there exists  $C > 1$  such that*

$$C^{-1} \cdot \text{Mod}_Q(\mathcal{F}) \leq \text{Mod}_Q(f(\mathcal{F})) \leq C \cdot \text{Mod}_Q(\mathcal{F})$$

*for any set of curves  $\mathcal{F}$  in  $X$ .*

The next proposition gives examples of spaces that do not satisfy the CLP.

**Proposition 2.17** ([HK98] or [BK13, Theorem 2.6.]). *Assume that  $Z$  satisfies the CLP then it has no local cut point, i.e. no connected open subset is disconnected by removing a point.*

Combining with the theorem of Bowditch (see [Bow98]) this proposition says that the boundary of a one-ended hyperbolic group that splits along a two-ended subgroup does not satisfy the CLP.

The first examples of spaces that satisfy the CLP are provided by Theorem 2.11 and by known examples of Loewner spaces. The next examples are provided by [BK13].

**Example 2.18.**

1. *The following spaces are Loewner spaces*
  - i) *the Euclidean space  $\mathbb{R}^d$  for  $d \geq 2$ , this result is due to C. Loewner for  $d \geq 3$  (see [Loe59]),*
  - ii) *any compact Riemannian manifold modeled by  $\mathbb{R}^d$  for  $d \geq 2$  (see [HK98]),*
  - iii) *visual boundaries of right-angled Fuchsian buildings (see [BP99]).*
2. *The following spaces satisfy the CLP (see [BK13])*
  - i) *the Sierpiński carpet and the  $n$ -dimensional Menger sponge embedded in the Euclidean space,*
  - ii) *boundaries of Coxeter groups of various type: simplex groups, some prism groups, some highly symmetric groups and some groups with planar boundary.*

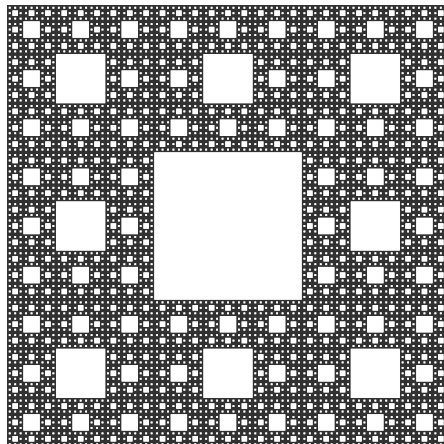


Figure 1: The Sierpiński carpet in  $\mathbb{E}^2$  satisfies the CLP

For Examples 2.18.2, we do not know if they are Loewner spaces. This provides a first kind of interesting questions.

**Question 2.19.** *Is any of the examples in 2.18.2. quasi-Moebius homeomorphic to a Loewner space?*

Among these examples the Sierpiński carpet is the one that should be studied first as it should be the easiest one.

Note that Example 2.18.1.ii) provides many examples of hyperbolic groups whose boundaries are Loewner spaces. Indeed consider a group  $\Gamma$  that is acting geometrically (see Subsection 3.1) on the standard hyperbolic space  $\mathbb{H}^d$  for  $d \geq 3$ . Then  $\partial\Gamma$  is quasi-Moebius equivalent to  $\mathbb{S}^{d-1}$  equipped with the standard spherical metric. Hence with Example 2.18.1.ii),  $\partial\Gamma$  is a Loewner space.

Now we consider the following general question: how the geometry of a hyperbolic space is determined by its boundary? The Cannon's conjecture (see Conjecture 1.2) is a question of this type. The notions of Loewner property and CLP have been fruitfully used by M. Bonk and B. Kleiner to approach this conjecture. By a theorem of D. Sullivan in [Sul82], Cannon's conjecture is equivalent to the following in which the quasiconformal structure of the boundary is the main point.

**Conjecture 2.20** ([BK02, Conjecture 1.3.]). *If  $\Gamma$  is a hyperbolic group and  $\partial\Gamma$  is homeomorphic to  $\mathbb{S}^2$ , then it is quasi-Moebius homeomorphic to the standard 2-sphere.*

If Conjecture 2.12 is true, the CLP would provide many interesting examples of Loewner spaces. This motivates our second question.

**Question 2.21.** *Can we find new examples of compact metric spaces satisfying the CLP?*

In this article, we exhibit examples of boundaries of right-angled buildings of dimension 3 and 4 that satisfy the CLP. These buildings have been studied by J. Dymara and D. Osajda who described the topology of the boundary.

**Theorem 2.22** ([DO07]). *Let  $\Delta$  be a right-angled thick building whose associated Coxeter group is a cocompact reflection group in  $\mathbb{H}^d$ . Then  $\partial\Delta$  is homeomorphic to the universal  $(d-1)$ -dimensional Menger space  $\mu^{d-1}$ .*

Our interest in these buildings is inspired by Examples 2.18.1.iii) and 2.18.2.ii). We will use some ideas from [BP00] and [BK13] where these examples are studied.

### 3 Combinatorial modulus of curve families on boundaries of hyperbolic groups

Boundaries of hyperbolic groups are naturally endowed with metric structures that satisfy a property of self-similarity. This property implies Proposition 3.11 that will be used several times in proving the main theorem. Intuitively, we can say that this proposition is a tool to enlarge sets of small curves while controlling the modulus.

In this section, we explain how the boundary of a hyperbolic group can be seen as approximately self-similar spaces. Then, we recall the connection between the combinatorial modulus and the conformal dimension of the boundary. Finally, we give a sufficient condition for the boundary to satisfy the CLP.

Most of this section is a review of [BK13, Section 3 and 4] to which we refer for details.

#### 3.1 Boundaries of hyperbolic groups and approximately self-similar spaces

For details concerning hyperbolic groups and spaces, we refer to [CDP90], [GdlH90] or [KB02]. Let  $(X, d)$  be a proper geodesic metric space. We say that a finitely generated group  $\Gamma$  *acts geometrically* on  $X$ , if:

- $\Gamma < \text{Isom}(X)$ ,
- $\Gamma$  acts cocompactly,
- $\Gamma$  acts properly discontinuously.

We say that  $X$  is *hyperbolic* (in the sense of Gromov) if there exists a constant  $\delta > 0$  such that for every geodesic triangle  $[x, y] \cup [y, z] \cup [z, x] \subset X$  and every  $p \in [x, y]$ , one has

$$\text{dist}(p, [y, z] \cup [z, x]) \leq \delta.$$

A finitely generated group that acts geometrically on a hyperbolic space  $X$  is called a *hyperbolic group*. In this case, the Cayley graph of a hyperbolic group is a hyperbolic space.

From now on, let  $X$  be a hyperbolic space with a fixed base point  $x_0$  and let  $\Gamma$  be a hyperbolic group acting geometrically on  $X$ . Let  $\partial X$  be the set of equivalence classes of geodesic rays where two geodesic rays  $\gamma, \gamma' : [0, +\infty) \rightarrow X$  are equivalent if and only if:

there exists  $K > 0$  such that  $d(\gamma(t), \gamma'(t)) \leq K$  for any  $t \in [0, +\infty)$ .

Thanks to the hyperbolicity condition, we can restrict to the set of geodesic rays starting from  $x_0$ . We can equip  $\partial X$  and  $X \cup \partial X$  with topologies which make them compact spaces. In this setting  $X$ , is dense in  $X \cup \partial X$  and  $\partial X$  is called the *boundary of  $X$* . Moreover we can equip  $\partial X$  with a family of *visual metric*. A metric  $\delta(\cdot, \cdot)$  is visual if there exist two constants  $A \geq 1$  and  $\alpha > 0$  such that for all  $\xi, \xi' \in \partial X$ :

$$A^{-1}e^{-\alpha\ell} \leq \delta(\xi, \xi') \leq A e^{-\alpha\ell},$$

where  $\ell$  is the distance from  $x_0$  to a geodesic line  $(\xi, \xi')$ . In such a situation we also write

$$\delta(\xi, \xi') \asymp e^{-\alpha\ell}.$$

The action of  $\Gamma$  on  $X$  extends naturally to  $(\partial X, \delta)$  and elements of  $\Gamma$  are uniform quasi-Moebius homeomorphisms of the boundary. Moreover, if  $\partial\Gamma$  is also equipped with a visual metric, the homeomorphism  $\partial\Gamma \rightarrow \partial X$  induced by the orbit map  $g \in \Gamma \rightarrow gx_0 \in X$  is quasi-Moebius.

The following definition is a generalization of the classical notion of self-similar space.

**Definition 3.1.** *A compact metric space  $(Z, d)$  is called approximately self-similar if there exists a constant  $L \geq 1$  such that for every  $r$ -ball  $B$  with  $0 < r < \text{diam } Z$ , there exists an open subset  $U \subset Z$  which is  $L$ -bi-Lipschitz homeomorphic to the rescaled ball  $(B, \frac{1}{r}d)$ .*

The definition and proposition that follow say that the boundary of a hyperbolic group is an approximately self-similar space and that this structure is linked to the action of the group on its boundary.

**Definition 3.2.** *Let  $\Gamma$  be a hyperbolic group. A metric  $d$  on  $\partial\Gamma$  is called a self-similar metric if there exists a hyperbolic space  $X$  on which  $\Gamma$  acts geometrically, such that  $d$  is the preimage of a visual metric on  $\partial X$  by the canonical quasi-Moebius homeomorphism  $\partial\Gamma \rightarrow \partial X$ .*

**Proposition 3.3** ([BK13, Proposition 3.3.]). *The space  $\partial\Gamma$  equipped with a self-similar metric is doubling and is an approximately self-similar space, the partial bi-Lipschitz maps being restrictions of group elements. Moreover,  $\Gamma$  acts on  $(\partial\Gamma, d)$  by (non-uniformly) bi-Lipschitz homeomorphisms.*



### 3.2 Combinatorial modulus and conformal dimension

In this subsection we present the connection between combinatorial modulus and the conformal dimension in approximately self-similar spaces.

Let  $Z$  be an arcwise connected approximately self-similar metric space. For us  $Z$  will be the boundary of a hyperbolic group. Let  $\{G_k\}_{k \geq 0}$  be a  $\kappa$ -approximation of  $Z$  and  $d_0$  be a small constant compared with  $\text{diam } Z$  and with the constant of self-similarity.

Let  $\mathcal{F}_0$  denote the set of all the curves in  $Z$  of diameter larger than  $d_0$ . In [BK13], it is proved that the properties of the combinatorial modulus are contained in the asymptotic behavior of  $\text{Mod}_p(\mathcal{F}_0, G_k)$ . This point is explained in Subsection 3.3. Here we write  $M_k := \text{Mod}_p(\mathcal{F}_0, G_k)$ .

The following proposition allows to define a critical exponent that is related to the conformal dimension of  $Z$ .

**Proposition 3.4.** *There exists  $p_0 \geq 1$  such that for  $p \geq p_0$  the modulus  $M_k$  goes to zero as  $k$  goes to infinity.*

*Proof.* Let  $\{G_k\}_{k \geq 0}$  be a  $\kappa$ -approximation of  $Z$ . According to the doubling condition and the definition of an approximation, there exists an integer  $N'$  such that each element  $v \in G_k$  is covered by at most  $N'$  elements of  $G_{k+1}$ . As a consequence, if  $K > 0$  is the cardinality  $G_0$ , then

$$\#G_k \leq K \cdot N'^k \text{ for any } k \geq 1.$$

Moreover, as we saw in the proof of Proposition 2.4, there exists a constant  $K' > 0$  such that the constant function  $\rho : v \in G_k \longrightarrow \rho(v) = K' \cdot 2^{-k} \in [0, +\infty)$  is  $\mathcal{F}_0$ -admissible.

As a consequence

$$M_k \leq C \cdot \left(\frac{N'}{2^p}\right)^k,$$

where  $C$  is a positive constant. Thus, for  $p$  large enough,  $M_k$  goes to zero.  $\square$

According to this proposition the next definition makes sense.

**Definition 3.5.** *The critical exponent  $Q$  associated with the curve family  $\mathcal{F}_0$  is defined as follows*

$$Q = \inf\{p \in [1, +\infty) : \lim_{k \rightarrow +\infty} \text{Mod}_p(\mathcal{F}_0, G_k) = 0\}.$$

We notice that for  $k \geq 0$  fixed the function  $p \longmapsto \text{Mod}_p(\mathcal{F}_0, G_k)$  is non-increasing. Hence  $\text{Mod}_p(\mathcal{F}_0, G_k)$  goes to zero for  $p > Q$ .

This critical exponent is related to the conformal dimension, which provides another motivation to study combinatorial modulus. The conformal dimension has been introduced by P. Pansu in [Pan89]. It is an important property of the conformal structure of the boundary of a hyperbolic group. In particular, it is invariant under quasi-Moebius maps.

In the following,  $\mathcal{H}_d(\cdot)$  and  $\dim_{\mathcal{H}}(Z, d)$  respectively denote the Hausdorff measure and the Hausdorff dimension of  $Z$  equipped with  $d$ . The *Ahlfors-regular conformal gauge* of  $(Z, d)$  is defined as follows

$$\mathcal{J}_c(Z, d) := \{(Z', \delta) : (Z', \delta) \text{ is AR and quasi-moebius homeomorphic to } (Z, d)\}.$$

**Definition 3.6.** *Let  $(Z, d)$  be a compact metric space. The Ahlfors-regular conformal dimension of  $(Z, d)$  is*

$$\text{Confdim}(Z, d) := \inf\{\dim_{\mathcal{H}}(Z', \delta) : (Z', \delta) \in \mathcal{J}_c(Z, d)\}.$$

In the rest of the paper we will simply call it the *conformal dimension*.

In [KK], S. Keith and B. Kleiner proved that the combinatorial modulus are related to the conformal dimension. The proof of the following theorem may be found in [Car11].

**Theorem 3.7** ([KK] or [Car11, Theorem 4.5.]). *The critical exponent  $Q$  (see Definition 3.5) is equal to  $\text{Confdim}(Z, d)$ .*

The definition of the conformal dimension, Combining with basic topology give the following inequalities:

$$\dim_T(Z) \leq \text{Confdim}(Z, d) \leq \dim_{\mathcal{H}}(Z, d),$$

where  $\dim_T(Z)$  is the topological dimension of  $Z$ .

The following theorem due to J. Tyson makes a connection between the conformal dimension and the Loewner property.

**Theorem 3.8** ([MT10, Corollary 4.2.2.]). *Let  $Q > 1$  and  $X$  be a  $Q$ -regular and  $Q$ -Loewner space, then  $\text{Confdim}(X) = Q$ .*

**Example 3.9.** *It has been proved independently by B. Kleiner and in [KL04] that the Euclidean metric on the Sierpiński carpet does not realize the conformal dimension. As a consequence the Sierpiński carpet equipped with this metric does not satisfy the Loewner property. However it satisfies the CLP (see Example 2.18).*

Again, Cannon's conjecture has been an important motivation for studying the conformal dimension of the boundary of a hyperbolic group. In particular in [BK05] it is proved that Conjecture 1.2 is equivalent to the following.

**Conjecture 3.10.** *If  $\Gamma$  is a hyperbolic group and  $\partial\Gamma$  is homeomorphic to  $\mathbb{S}^2$ , then  $\text{Confdim}(\partial\Gamma)$  is attained by a metric in  $\mathcal{J}_c(\partial\Gamma)$ .*

### 3.3 How to prove the CLP

Now we give the sufficient condition that will be used in this article to exhibit some examples of groups with a boundary that satisfies the CLP.

Let  $Z$  be an arcwise connected approximately self-similar metric space and let  $\{G_k\}_{k \geq 0}$  be a  $\kappa$ -approximation of  $Z$ .

The following proposition says that the combinatorial modulus are preserved by the bi-Lipschitz homeomorphism coming from the approximately self-similar structure. This proposition will be used several times in Sections 6 to 10 to compare the modulus of a set of small curves with the modulus of a set of curves of diameter larger than a fixed constant.

**Proposition 3.11.** *Let  $B$  be a ball in  $\partial\Gamma$  such that  $\text{diam } B < 1$ . Let  $g \in \Gamma$  be the local  $L$ -bi-Lipschitz homeomorphism that rescales  $B$  (given by Definition 3.3). Let  $\mathcal{F}$  be a set of curves contained in  $\lambda B$  for  $\lambda < 1$ . Then there exist  $\ell \in \mathbb{N}$ , and  $D > 1$  depending only on  $L$  and on the doubling constant of  $\partial\Gamma$  such that the following property holds.*

*If  $k \geq 0$  is large enough so that*

$$\{v \in G_k : \gamma \cap v \neq \emptyset \text{ for some } \gamma \in \mathcal{F}\} \subset \{v \in G_k : v \subset B\},$$

*then*

$$D^{-1} \cdot \text{Mod}_p(g\mathcal{F}, G_k) \leq \text{Mod}_p(\mathcal{F}, G_{k+\ell}) \leq D \cdot \text{Mod}_p(g\mathcal{F}, G_k),$$

*where  $g\mathcal{F} = \{g\gamma : \gamma \in \mathcal{F}\}$ .*

*Proof.* Let  $k \geq 0$  be large enough so that, if  $\gamma \cap v \neq \emptyset$  with  $\gamma \in \mathcal{F}$  and  $v \in G_k$ , then  $v \subset B$ . Let  $d = \text{diam } B$  and let  $\ell \in \mathbb{N}$  denote the integer satisfying  $2^{-(\ell+1)} < d \leq 2^{-\ell}$ . Let  $v \in G_{k+\ell}$  such that  $v \subset B$  and assume

$$B(\xi, \kappa^{-1}2^{-(k+\ell)}) \subset v \subset B(\xi, \kappa 2^{-(k+\ell)}).$$

Then

$$B(g\xi, (L\kappa)^{-1}2^{-k}) \subset gv \subset B(g\xi, 4L\kappa 2^{-k}).$$

We write  $G'_k \cap gB = \{gv : v \in G_{k+\ell}, v \subset B\}$ . Then  $G'_k \cap gB$  is a  $\kappa'$ -approximation of  $gB$  on scale  $k$ , with  $\kappa'$  depending only on  $\kappa$  and  $L$ . As the curves of  $\mathcal{F}$  are strictly contained in  $B$  we obtain the following equality

$$\text{Mod}_p(\mathcal{F}, G_{k+\ell}) = \text{Mod}_p(g\mathcal{F}, G'_k \cap gB).$$

Thanks to Proposition 2.5, there exists  $D > 1$  such that

$$D^{-1} \cdot \text{Mod}_p(g\mathcal{F}, G_k) \leq \text{Mod}_p(g\mathcal{F}, G'_k \cap gB) \leq D \cdot \text{Mod}_p(g\mathcal{F}, G_k),$$

and the proposition follows.  $\square$

Now we fix  $d_0 > 0$  a small constant compared with  $\text{diam } Z$  and with the constant of self-similarity. More precisely it must be small enough so that any non-constant curve in  $Z$  may be rescaled to a curve of diameter larger than  $d_0$  by self-similarity. For us,  $Z$  will be the boundary of a hyperbolic group and  $d_0$  will depend on the hyperbolicity constant. In the following,  $\mathcal{F}_0$  is the set of all the curves in  $Z$  of diameter larger than  $d_0$ .

Again, we use the letter  $Q$  to designate the critical exponent of Definition 3.5. We recall that if  $\eta$  is a curve in  $Z$ , then  $\mathcal{U}_\epsilon(\eta)$  denotes the  $\epsilon$ -neighborhood of  $\eta$  for the  $C^0$ -topology. This means that a curve  $\eta' \in \mathcal{U}_\epsilon(\eta)$  if and only if there exists  $s : t \in [0, 1] \rightarrow [0, 1]$  a parametrization of  $\eta$  such that for any  $t \in [0, 1]$  one has  $d(\eta(s(t)), \eta'(t)) < \epsilon$ .

The following proposition gives the sufficient conditions for  $Z$  to satisfy the CLP that will be used in proving the main theorem.

**Proposition 3.12** ([BK13, Proposition 4.5.]). *Let  $Z$  be an arcwise connected approximately self-similar metric space. Let  $\{G_k\}_{k \geq 0}$  be a  $\kappa$ -approximation of  $Z$  and  $d_0$  be a small constant compared with  $\text{diam } Z$  and with the constant of self-similarity. Let  $\mathcal{F}_0$  denote the set of all the curves in  $Z$  of diameter larger than  $d_0$ .*

*For  $p = 1$ , we assume that  $\text{Mod}_p(\mathcal{F}_0, G_k)$  is unbounded. For  $p \geq 1$ , we assume that for every non-constant curve  $\eta \subset Z$  and every  $\epsilon > 0$ , there exists  $C = C(p, \eta, \epsilon)$  such that for every  $k \in \mathbb{N}$ :*

$$\text{Mod}_p(\mathcal{F}_0, G_k) \leq C \cdot \text{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

*Suppose furthermore when  $p$  belongs to a compact subset of  $[1, +\infty)$  the constant  $C$  may be chosen independent of  $p$ . Then  $Z$  satisfies the combinatorial  $Q$ -Loewner property.*

## 4 Steps in the proof of Theorem 10.1

Before going into details about boundaries of right-angled buildings, we give a sketch of proof of the main theorem. In this section,  $D$  is the right-angled dodecahedron in  $\mathbb{H}^3$  or the right-angled 120-cell in  $\mathbb{H}^4$ . We write  $W_D$  for the hyperbolic reflection group generated by the faces of  $D$ . The main theorem of this article may be stated as follows.

**Theorem 4.1** (Corollary 10.3). *Let  $\Gamma$  be the graph product of constant thickness  $q \geq 3$  and of Coxeter group  $W_D$ . Then  $\partial\Gamma$  equipped with a visual metric satisfies the CLP.*

As announced, we will verify that  $\partial\Gamma$  satisfies the hypothesis of Proposition 3.12. To prove that  $\text{Mod}_1(\mathcal{F}_0, G_k)$  is unbounded, it is enough to see that for every  $N \in \mathbb{N}$  there exist  $N$  disjoint curves in  $\partial\Gamma$ . Indeed, this implies that for  $k \geq 0$  large enough  $\text{Mod}_1(\mathcal{F}_0, G_k) > N$ .

**To follow curves to control the modulus.** For  $p > 1$ , we want to prove that the curves of  $\partial\Gamma$  satisfy the following property.

(P) : For  $\epsilon > 0$ , there exists a finite set  $F$  of bi-Lipschitz maps  $f : \partial\Gamma \rightarrow \partial\Gamma$  such that for any curve  $\gamma \in \mathcal{F}_0$  and any curve  $\eta$  in  $\partial\Gamma$ , the subset  $\bigcup_{f \in F} f(\gamma)$  of  $\partial\Gamma$  contains a curve that belongs to  $\mathcal{U}_\epsilon(\eta)$ .

Where  $\mathcal{U}_\epsilon(\eta)$  denotes the  $\epsilon$ -neighborhood of  $\eta$  for the  $C^0$  distance (see Subsection 1.4 for details). Intuitively,  $(P)$  holds if from any curve  $\gamma$  we can *follow* any other curve  $\eta$  using bi-Lipschitz maps. The following computation shows that property  $(P)$  implies the desired inequality.

**Proposition 4.2.** *If  $\text{Mod}_1(\mathcal{F}_0, G_k)$  is unbounded, then property  $(P)$  implies the CLP.*

*Proof.* Let  $\eta$  be a curve in  $\partial\Gamma$  and  $\epsilon > 0$ . Fix  $\rho$  a  $\mathcal{U}_\epsilon(\eta)$ -admissible function. The inequality required by the hypothesis of Proposition 3.12 is obtained if we find a constant  $K > 0$  independent of the scale  $k$  and a  $\mathcal{F}_0$ -admissible function  $\rho'$  such that  $M_p(\rho') \leq K \cdot M_p(\rho)$ .

Let  $F$  be the set of bi-Lipschitz maps given by the property  $(P)$ . We assume, without loss of generality that  $F$  contains  $F^{-1}$ . We define the function  $\rho' : G_k \rightarrow [0, +\infty)$  by:

$$(*) \quad \rho'(v) = \sum_{f \in F} \sum_{fw \cap v \neq \emptyset} \rho(w).$$

Let  $\gamma \in \mathcal{F}_0$  and  $\theta$  be a curve contained in  $\bigcup_{f \in F} f\gamma \subset \partial\Gamma$  such that  $\theta \in \mathcal{U}_\epsilon(\eta)$ . Then

$$L_{\rho'}(\gamma) = \sum_{f \in F} \sum_{v \cap \gamma \neq \emptyset} \sum_{fw \cap v \neq \emptyset} \rho(w) \geq \sum_{f \in F} \sum_{w \cap f\gamma \neq \emptyset} \rho(w).$$

On the other hand

$$L_\rho(\theta) \leq \sum_{f \in F} L_\rho(f\gamma) = \sum_{f \in F} \sum_{w \cap f\gamma \neq \emptyset} \rho(w).$$

Hence  $L_{\rho'}(\gamma) \geq L_\rho(\theta)$  and  $\rho'$  is  $\mathcal{F}_0$ -admissible.

Then the number of terms in the right-hand side of the definition  $(*)$  is bounded by a constant  $N$  depending only on  $\#F$ , the bi-Lipschitz constants of the elements of  $F$ , and the doubling constant of  $\partial\Gamma$ . Therefore by convexity

$$M_p(\rho') = \sum_{v \in G_k} \left( \sum_{f \in F} \sum_{fw \cap v \neq \emptyset} \rho(w) \right)^p \leq N^{p-1} \cdot \sum_{v \in G_k} \sum_{f \in F} \sum_{w \cap fv \neq \emptyset} \rho(w)^p \leq N^p \cdot \#F \cdot M_p(\rho).$$

□

Note that this idea of following curves may be used to obtain an inequality between the modulus of any pair of sets of curves.

**The issue of parabolic limit sets.** Since  $\Gamma$  acts on  $\partial\Gamma$  by bi-Lipschitz homeomorphisms, it is natural to try to follow curves by using the elements of  $\Gamma$ . However, in right-angled buildings some curves may be contained in parabolic limit sets (boundaries of residues). As we will see in Example 6.8, these curves are obstacles to proving the property  $(P)$  by using the elements of  $\Gamma$ .

To solve this problem we start by showing that  $\text{Mod}_p(\mathcal{F}_0, G_k)$  is determined by the combinatorial modulus of the sets of all the curves contained in a parabolic limit set. This is done at the beginning of the proof of Theorem 8.13.

**Following curves inside parabolic limit sets.** Then inside the parabolic limit set  $\partial P$  it is possible to follow curves. An analogue of property (P) inside the parabolic limit sets is proved in Proposition 6.11. From this property we can obtain Theorem 6.12. This theorem is the first major step toward the proof. Essentially it says that the combinatorial modulus of  $\mathcal{F}_{\delta,r}(\partial P)$  is controlled by any curve contained in  $\partial P$ .

**Controlling the modulus in  $\partial\Gamma$  by the modulus in the boundary of an apartment.**

The second major step in the proof is to use the building structure to reduce the problem in  $\partial\Gamma$  to a problem in the boundary of an apartment *i.e* in  $\partial W_D$ . This is done by Theorem 8.9. Essentially, this theorem says that the modulus of a curve family in  $\partial\Gamma$  is controlled by a weighted modulus defined in the boundary of an apartment. The idea used to prove this is that, from the point of view of the modulus,  $\partial\Gamma$  can be seen as the direct product of the boundary of an apartment by a finite set whose cardinality depends on the scale.

**Conclusion of the proof thanks to the symmetries of  $D$ .** Now, thanks to Theorems 6.12 and 8.9, we arrive at a point where the modulus of  $\mathcal{F}_{\delta,r}(\partial P)$ , and thus of  $\mathcal{F}_0$ , is controlled by some modulus of the parabolic limit sets of  $W_D$ . The idea we use to conclude is that the symmetries of  $D$  extends to the boundary of  $W_D$ . Combining with the elements of the groups, these symmetries permit us to follow curves in  $\partial W_D$ . As a consequence we obtain a strong control of the modulus of the parabolic limit sets in  $\partial W_D$  and we can complete the proof.

## 5 Locally finite right-angled hyperbolic buildings

The aim of this article is to study combinatorial modulus on boundaries of hyperbolic buildings. Below we set up the context about hyperbolic buildings that will be used until the end of this article. In particular, we discuss the geometry of locally finite right-angled hyperbolic buildings.

For details concerning the notions recalled in this section, we refer to [Tit74], [Ron89], or [AB08]. Concerning the Davis realization, we can refer to [Dav08, Chapter 8] or to [Mei96] for an example of the Davis construction along with suggestive pictures. Below  $S = \{s_1, \dots, s_n\}$  is a fixed finite set.

### 5.1 Chamber systems

Following the definition of J. Tits, a *chamber system*  $X$  over  $S$  is a set endowed with a family of partitions indexed by  $S$ . The elements of  $X$  are called *chambers*.

Hereafter  $X$  is a chamber system over  $S$ . For  $s \in S$ , two chambers  $c, c' \in X$  are said to be *s-adjacent* if they belong to the same subset of  $X$  in the partition associated with  $s$ . Then we write  $c \sim_s c'$ . Usually, omitting the type of adjacency we refer to *adjacent* chambers and we write  $c \sim c'$ . Note that any chamber is adjacent to itself.

A *morphism*  $f : X \rightarrow X'$  between two chamber systems  $X, X'$  over  $S$  is a map that preserves the adjacency relations. A bijection of  $X$  that preserves the adjacency relations is called an *automorphism* and we designate by  $\text{Aut}(X)$  the *group of automorphisms* of  $X$ . A *subsystem of chamber*  $Y$  of  $X$  is a subset  $Y \subset X$  such that the inclusion map is a morphism of chamber systems.

We call *gallery*, a finite sequence  $\{c_k\}_{k=1,\dots,\ell}$  of chambers such that  $c_k \sim c_{k+1}$  for  $k = 1, \dots, \ell - 1$ . The galleries induce a *metric* on  $X$ .

**Definition 5.1.** *The distance between two chambers  $x$  and  $y$  is the length of the shortest gallery connecting  $x$  to  $y$ .*

We use the notation  $d_c(\cdot, \cdot)$  for this metric over  $X$ . A shortest gallery between two chambers is called *minimal*.

Let  $I \subset S$ . A subset  $C$  of  $X$  is said to be *I-connected* if for any pair of chambers  $c, c' \in C$  there exists a gallery  $c = c_1 \sim \dots \sim c_\ell = c'$  such that for any  $k = 1, \dots, \ell - 1$ , the chambers  $c_k$  and  $c_{k+1}$  are  $i_k$ -adjacent for some  $i_k \in I$ .

**Definition 5.2.** *The I-connected components are called the I-residues or the residues of type I. The cardinality of I is called the rank of the residues of type I. The residues of rank 1 are called panels.*

The following notion of convexity is used in chamber systems.

**Definition 5.3.** *A subset  $C$  of  $X$  is called convex if every minimal gallery whose extremities belong to  $C$  is entirely contained in  $C$ .*

The convexity is stable by intersection and for  $A \subset X$ , the *convex hull* of  $A$  is the smallest convex subset containing  $A$ . In particular, convex subsets of  $X$  are subsystems. The following example is crucial because it will be used to equip Coxeter groups and graph products with structures of chamber systems (see Definition 5.7 and Theorem 5.16).

**Example 5.4.** *Let  $G$  be a group,  $B$  a subgroup and  $\{H_i\}_{i \in I}$  a family of subgroups of  $G$  containing  $B$ . The set of left cosets of  $H_i/B$  defines a partition of  $G/B$ . We denote by  $C(G, B, \{H_i\}_{i \in I})$  this chamber system over  $I$ . This chamber system comes with a natural action of  $G$ . The group  $G$  acts by automorphisms and transitively on the set of chambers.*

## 5.2 Coxeter systems

A *Coxeter matrix* over  $S$  is a symmetric matrix  $M = \{m_{r,s}\}_{r,s \in S}$  whose entries are elements of  $\mathbb{N} \cup \{\infty\}$  such that  $m_{s,s} = 1$  for any  $s \in S$  and  $\{m_{r,s}\} \geq 2$  for any  $r, s \in S$  distinct. Let  $M$  be a Coxeter matrix. The *Coxeter group* of type  $M$  is the group given by the following presentation

$$W = \langle s \in S \mid (rs)^{m_{r,s}} = 1 \text{ for any } r, s \in S \rangle.$$

We call *special subgroup* a subgroup of  $W$  of the form

$$W_I = \langle s \in I \mid (rs)^{m_{r,s}} = 1 \text{ for any } r, s \in I \rangle \text{ with } I \subset S.$$

**Definition 5.5.** We call parabolic subgroup a subgroup of  $W$  of the form  $wW_Iw^{-1}$  where  $w \in W$  and  $I \subset S$ . An involution of the form  $ws w^{-1}$  for  $w \in W$  and  $s \in S$  is called a reflection.

**Example 5.6.** Let  $\mathbb{X}^d = \mathbb{S}^d, \mathbb{E}^d$  or  $\mathbb{H}^d$ . A Coxeter polytope is a convex polytope of  $\mathbb{X}^d$  such that any dihedral angle is of the form  $\frac{\pi}{k}$  with  $k$  not necessarily constant. Let  $D$  be a Coxeter polytope and let  $\sigma_1, \dots, \sigma_n$  be the codimension 1 faces of  $D$ . We set  $M = \{m_{i,j}\}_{i,j=1,\dots,n}$  the matrix defined by  $m_{i,i} = 1$ ,  $m_{i,j} = \infty$  if  $\sigma_i$  and  $\sigma_j$  do not meet in a codimension 2 face, and  $m_{i,j} = k$  if  $\sigma_i$  and  $\sigma_j$  meet in a codimension 2 face and  $\frac{\pi}{k}$  is the dihedral angle between  $\sigma_i$  and  $\sigma_j$ .

Then a theorem of H. Poincaré (see [GP01, Theorem 1.2.]) says that the reflection group of  $\mathbb{X}^d$  generated by the codimension 1 faces of  $D$  is a discrete subgroup of  $\text{Isom}(\mathbb{X}^d)$  and is isomorphic to the Coxeter group of type  $M$ .

**Definition 5.7.** With the notation introduced in Example 5.4, the Coxeter system associated with  $W$  is the chamber system over  $S$  given by  $C(W, \{e\}, \{W_{\{s\}}\}_{s \in S})$ . We use the notation  $(W, S)$  to denote this chamber system.

The chambers of  $(W, S)$  are the elements of  $W$  and two distinct chambers  $w, w' \in W$  are  $s$ -adjacent if and only if  $w = w's$ . For  $I \subset S$ , notice that for any  $I$ -residue  $R$  in  $(W, S)$  there exists  $w \in W$  such that, as a set  $R = wW_I$ . Again  $W$  is a group of automorphisms of  $(W, S)$  that acts transitively on the set of chambers.

Hereafter  $(W, S)$  is a fixed Coxeter system.

**Example 5.8.** In the case of Example 5.6, the chamber system associated with  $W$  is realized geometrically by the tiling of  $\mathbb{X}^d$  by copies of the polytope  $D$ . Two chambers are adjacent in  $(W, S)$  if and only if the corresponding copies of  $D$  in  $\mathbb{X}^d$  share a codimension 1 face.

### 5.3 The Davis chamber of $(W, S)$

To a Coxeter group  $W$ , M.W. Davis associates a cellular complex  $D$  called the *Davis chamber*. In the particular case of reflection groups (see Examples 5.6 and 5.8) the Davis Chamber is the Coxeter polytope.

We recall that  $S = \{s_1, \dots, s_n\}$  is a set of generators of  $W$ . Let  $\mathcal{S}_{\neq S}$  be the collection of proper subsets of  $S$ . We denote by

$\mathcal{S}_f$  the set of proper subsets  $F \subsetneq S$  such that  $W_F$  is finite.

By [Dav08, Appendix A], a poset admits a geometric realization which is a simplicial complex. This complex is such that the inclusion relations between cells represent the partial order. We denote by  $D$  the *Davis chamber* which is the geometric realization of the poset  $\mathcal{S}_f$ . In the following we give details of this construction.

Let  $\Delta^{n-1}$  be the standard  $(n-1)$ -simplex and label the codimension 1 faces of  $\Delta^{n-1}$  with distinct elements of  $S$ . Then a codimension  $k$  face  $\sigma$  of  $\Delta^{n-1}$  is associated with a



type i.e a subset  $I \subset S$  of cardinality  $k$ . In this case, we write  $\sigma_I$  for the *face of type  $I$* . Equivalently, we can say that each vertex of the barycentric subdivision of  $\Delta^{n-1}$  is associated with a subset of  $S$ . Combining with the fact that the empty set is associated with the barycenter of the whole simplex, we get a bijection between the vertices of the barycentric subdivision and  $\mathcal{S}_{\neq S}$ . Hence a vertex in the barycentric subdivision is designated by  $(s_i)_{i \in K}$  for  $K \subset \{1, \dots, n\}$ . Using this identification, let  $\mathcal{T}$  be the subgraph of the 1-skeleton of the barycentric subdivision of  $\Delta^{n-1}$  defined as follows:

- $\mathcal{T}^{(0)} = \mathcal{S}_{\neq S}$ ,
- the vertices  $(s_i)_{i \in I}$  and  $(s_j)_{j \in J}$ , with  $\#J \geq \#I$ , are adjacent if and only if  $I \subset J$  and  $\#I = \#J - 1$ .

In the following definition, for  $k \geq 1$  we call a  $k$ -cube, a CW-complex that is isomorphic, as a cellular complex, to the Euclidean  $k$ -cube  $[0, 1]^k$ . In particular, it is not necessary to equip these cubes with a metric for the purpose of this chapter.

**Definition 5.9.** *The 1-skeleton  $D^{(1)}$  of the Davis chamber is the full subgraph of  $\mathcal{T}$  generated by the elements of  $\mathcal{S}_f$ . The Davis chamber is obtained from  $D^{(1)}$  by attaching a  $k$ -cube to every subgraph that is isomorphic to the 1-skeleton of a  $k$ -cube.*

By construction,  $D \subset \Delta^{n-1}$ . We call *maximal faces* of  $D$  the subsets of the form  $\sigma \cap D$  where  $\sigma$  is a codimension 1 face of  $\Delta^{n-1}$ . Likewise, for  $I \subset S$ , the *face of  $D$  of type  $I$*  is  $D \cap \sigma_I$ .

**Example 5.10.** *In the case of Example 5.6, the Davis chamber is combinatorially identified with the Coxeter polytope. So if we equip  $D$  with the appropriate metric (Euclidean, spherical, or hyperbolic) we recover the Coxeter polytope.*

## 5.4 Buildings

Buildings are singular spaces defined by J. Tits. We may view them as higher dimensional analogues of trees. Hereafter  $(W, S)$  is a fixed Coxeter system.

**Definition 5.11** ([Tit74, Definition 3.1.]). *A chamber system  $\Delta$  over  $S$  is a building of type  $(W, S)$  if it admits a maximal family  $\mathcal{Ap}(\Delta)$  of subsystems isomorphic to  $(W, S)$ , called apartments, such that*

- *any two chambers lie in a common apartment,*
- *for any pair of apartments  $A$  and  $B$ , there exists an isomorphism from  $A$  to  $B$  fixing  $A \cap B$ .*

An immediate consequence of this definition is the existence of retraction maps of the building onto the apartments.

**Definition 5.12.** Let  $x \in \Delta$  and  $A \in \mathcal{Ap}(\Delta)$ . Assume that  $x$  is contained in  $A$ . We call retraction onto  $A$  centered at  $x$  the map  $\pi_{A,x} : \Delta \longrightarrow A$  defined by the following property.

For  $c \in \Delta$ , there exists a chamber  $\pi_{A,x}(c) \in A$  such that for any apartment  $A'$  containing  $x$  and  $c$ , for any isomorphism  $f : A' \longrightarrow A$  that fixes  $A \cap A'$ , we have  $f(c) = \pi_{A,x}(c)$

Hereafter,  $\Delta$  is a fixed building of type  $(W, S)$ . The building  $\Delta$  is called a *thin* (resp. *thick*) building if any panel contains exactly two (resp. at least three) chambers. Note that thin buildings are Coxeter systems.

## 5.5 Graph products and right-angled buildings

Let  $\mathcal{G}$  denote a *finite simplicial graph* i.e  $\mathcal{G}^{(0)}$  is finite, each edge has two different vertices, and  $\mathcal{G}$  contains no double edge. We denote by  $\mathcal{G}^{(0)} = \{v_1, \dots, v_n\}$  the vertices of  $\mathcal{G}$ . If for  $i \neq j$ , the corresponding vertices  $v_i, v_j$  are connected by an edge, we write  $v_i \sim v_j$ . A finite cyclic group  $G_i = \langle s_i \rangle$  of order  $q_i \geq 2$  is associated with each  $v_i \in \mathcal{G}^{(0)}$  and we set  $S = \{s_1, \dots, s_n\}$ . Throughout this article, we assume that  $n \geq 2$  and that  $\mathcal{G}$  has at least one edge.

**Definition 5.13.** The graph product given by  $(\mathcal{G}, \{G_i\}_{i=1, \dots, n})$  is the group defined by the following presentation

$$\Gamma = \langle s_i \in S \mid s_i^{q_i} = 1, s_i s_j = s_j s_i \text{ if } v_i \sim v_j \rangle.$$

**Example 5.14.** If the graph  $\mathcal{G}$  is fixed with  $q_i \in \{2, 3, \dots\}$ , graph products are groups between right-angled Coxeter groups (see [Dav08]) and right-angled Artin groups (see [Cha07]). If the integers  $\{q_i\}_{i=1, \dots, n}$  are fixed and we add edges to the graph starting from a graph with no edge, those groups are groups between free products and direct products of cyclic groups.

From now on, we fix a graph product  $\Gamma$  associated with the pair  $(\mathcal{G}, \{G_i\}_{i=1, \dots, n})$ . By analogy with Definition 5.5, we define parabolic subgroups in  $\Gamma$ .

**Definition 5.15.** The subgroup of  $\Gamma$  generated by a subset  $I \subset S$  is denoted by  $\Gamma_I$  and a subgroup of the form  $g\Gamma_I g^{-1}$ , with  $g \in \Gamma$ , is called a parabolic subgroup.

Let  $W$  be the graph product defined by the pair  $(\mathcal{G}, \{\mathbb{Z}/2\mathbb{Z}\}_{i=1, \dots, n})$ . This graph product is isomorphic to the right-angled Coxeter group of type  $M = \{m_{i,j}\}_{i,j=1, \dots, n}$  defined by :  $m_{i,j} = 2$  if  $v_i \sim v_j$  and  $m_{i,j} = \infty$  if  $v_i \not\sim v_j$ .

Throughout this article,  $W$  denotes this Coxeter group canonically associated with  $\Gamma$  and  $(W, S)$  is the Coxeter system associated with  $W$ .

**Theorem 5.16** ([Dav98, Theorem 5.1.]). Let  $\Delta$  be the chamber system  $C(\Gamma, \{e\}, \{\Gamma_{\{s\}}\}_{s \in S})$  (see Example 5.4). Then  $\Delta$  is a building of type  $(W, S)$ .

Hereafter,  $\Delta$  denotes the right-angled building associated with  $\Gamma$  by the preceding theorem. In Subsection 5.6, we describe the Davis complex associated with this building.

We notice that  $\Gamma$  is infinite if and only if the graph  $\mathcal{G}$  contains two distinct vertices  $v_i, v_j$  such that  $v_i \approx v_j$ . A criterion of M. Gromov allows J. Meier to prove that an infinite graph product  $\Gamma$  is hyperbolic if and only if in  $\mathcal{G}$  any circuit of length 4 contains a chord (see [Mei96]). For an infinite hyperbolic graph product, a necessary and sufficient condition is given in [DM02] for  $\partial\Gamma$  to be arcwise connected (see Theorem 6.16 below). This condition involves only the graph  $\mathcal{G}$ . In the rest of this paper, we will assume that  $\Gamma$  is infinite hyperbolic with arcwise connected boundary  $\partial\Gamma$ .

## 5.6 The Davis complex associated with $\Gamma$

To a graph product  $\Gamma$ , M.W. Davis associates a cellular complex  $\Sigma$  called the *Davis complex*. This complex is a metric space on which  $\Gamma$  acts geometrically. In the particular case of reflection groups (see Examples 5.6 and 5.8) the Davis Complex is  $\mathbb{X}^d$  tiled by the Coxeter polytopes. This complex is also called the geometric realization of the building  $\Delta$ . Below we introduce the Davis complex associated with  $\Gamma$ . Again we refer to [Mei96] for an example along with suggestive pictures.

Let  $D$  be the Davis chamber associated with  $W$  as in Subsection 5.3. Again a face of  $D$  is associated with a type  $I \subset S$ . For  $x \in D$ , if  $I$  is the type of the face containing  $x$  in its interior, we set  $\Gamma_x := \Gamma_I$ . To the interior points of  $D$  we associate the trivial group  $\Gamma_\emptyset$ .

Now we can define the *Davis complex* :  $\Sigma = D \times \Gamma / \sim$  with

$$(x, g) \sim (y, g') \text{ if and only if } x = y \text{ and } g^{-1}g' \in \Gamma_x.$$

We study the building  $\Delta$  through its geometric realization  $\Sigma$  and we briefly recall what this means.

A chamber of  $\Sigma$  is a subset of the form  $[D \times \{g\}]$  with  $g \in \Gamma$ . Two chambers are adjacent if and only if they share a maximal face. For a subset  $E \subset \Sigma$  we designate by  $\text{Ch}(E)$  the set of chambers contained in  $E$ . Equipped with this chamber system structure,  $\Sigma$  is isomorphic to  $\Delta$ . In particular, the set of apartments in  $\Sigma$  is designated by  $\mathcal{Ap}(\Sigma)$ . Then the left action of  $\Gamma$  on itself induces an action on  $\Sigma$ . For  $\gamma \in \Gamma$  and  $[(x, g)] \in \Sigma$  we set  $\gamma[(x, g)] := [(\gamma x, \gamma g)]$ . Moreover this action induces a simply transitive action of  $\Gamma$  on  $\text{Ch}(\Sigma)$ . Naturally this action is also isometric for  $d_c(\cdot, \cdot)$ .

**Example 5.17.** *In the case of Example 5.6, if we equip  $D$  with the appropriate metric we see that the Davis complex is realized by the tiling of  $\mathbb{X}^d$  by  $D$ .*

## 5.7 Building-walls and residues in the Davis complex

We call *base chamber* of  $\Sigma$ , denoted by  $x_0$ , the chamber  $[D \times \{e\}]$ . For  $g \in \Gamma$ , as  $[D \times \{g\}]$  is the image of  $x_0$  under  $g$ , we designate this chamber by  $gx_0$ . Below we present some basic

tools used to describe the structure of  $\Sigma$ . In particular, we extend to  $\Sigma$  some definitions and properties that have been used in Coxeter systems.

The notion of walls in a Coxeter system extends to right-angled buildings.

**Definition 5.18.**

1. We call *building-wall* in  $\Sigma$  the subcomplex  $M$  stabilized by a non-trivial isometry  $r = gs^\alpha g^{-1}$  with  $g \in \Gamma$ ,  $s \in S$ ,  $\alpha \in \mathbb{Z}$  and  $s^\alpha \neq e$ . The isometry  $r$  is called a *rotation* around  $M$ . We denote by  $\mathcal{M}(\Sigma)$  the set of all the building-walls of  $\Sigma$ .
2. Let  $M$  be a building-wall associated with a rotation  $r \in \Gamma$ . For  $x \in \text{Ch}(\Sigma)$  we say that  $M$  is *along*  $x$  if  $r(x)$  is adjacent to  $x$ .

With the notation of the definition, the term  $s \in S$  is called the *type of the building-wall*  $M$  and of the rotation  $r$ . We remark that in the graph product  $\Gamma$  two elements of  $S$  that are conjugate are equal so the type is uniquely defined. Indeed if  $s = gs'g^{-1}$  with  $g \in \Gamma$ ,  $s, s' \in S$ , let  $M$  be the building-wall associated with the rotation  $s$ . Then we observe that from the base chamber  $x_0$  we can reach the chamber  $gx_0$  by successive rotations about faces that are all orthogonal to  $M$ . In other word  $g$  is a product of elements of  $S$  that all commute with  $s$ . Thus  $s = s'$ . Clearly with the notation of the definition, the building-wall  $M$  is fixed by any rotation  $gs^{\alpha'}g^{-1}$  with  $s^{\alpha'} \neq e$ .

We say that the building-wall  $M$  is non-trivial if it contains more than one point. A non-trivial building-wall  $M$  may be equipped with a building structure. Indeed, if  $s_i$  is the type of  $M$ , associated with  $v_i \in \mathcal{G}^{(0)}$ , we write  $I = \{j : v_j \sim v_i, v_j \neq v_i\}$  and  $V = \{v_j \in \mathcal{G}^{(0)} : j \in I\}$ . Then if  $\mathcal{G}_V$  is the full subgraph generated by  $V$ , we can check that,  $M$  is isomorphic to the geometric realization of the graph product  $(\mathcal{G}_V, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i \in I})$ . The Davis chamber of this geometric realization is the maximal face of type  $s_i$  of  $D$ . Moreover building-walls also divide  $\Sigma$  in isomorphic connected components. In the case of infinite dimension 2 buildings, the building-walls are trees and thus they have been called *trees-walls* by M. Bourdon and H. Pajot in [BP00]. These explain our terminology.

**Definition 5.19.** Let  $M$  be a building-wall of type  $s$  and let  $r \in \Gamma$  be a rotation around  $M$ . A *dial* of building bounded by  $M$  is the closure in  $\Sigma$  of a connected component of  $\Sigma \setminus M$ . We denote by  $\mathcal{D}(\Sigma)$  the set of all the dials of building of  $\Sigma$ .

This definition implies the following fact.

**Fact 5.20.** Let  $M$  be a building-wall of type  $s$ . Assume that  $s$  is of finite order  $q$ . Then  $\Sigma \setminus M$  consists of  $q$  connected components. We designate by  $D_0(M), D_1(M), \dots, D_{q-1}(M)$  these dials of building, with the convention that  $x_0 \subset D_0(M)$ . In this setting, for any  $i = 0, \dots, q-1$ , if  $y \in \text{Ch}(D_i(M))$  then

$$\text{Ch}(D_i(M)) = \{x \in \text{Ch}(\Sigma) : d_c(y, x) < d_c(y, rx)\}.$$

Finally  $r$  permutes  $D_0(M), D_1(M), \dots, D_{q-1}(M)$ .

For a building-wall associated with a type  $s \in S$  of infinite order, the analogous property holds.

In thin right-angled buildings, in particular in apartments, building-walls are called *walls* and dials of building are called *half-spaces*. For  $A \in \mathcal{Ap}(\Delta)$  we write  $\mathcal{M}(A)$  for the set of all the walls and  $\mathcal{H}(A)$  for the set of all the half-spaces of  $A$ .

The building-walls in  $\Sigma$  and the walls in the apartments are closely related.

**Fact 5.21.** *i) For  $A \in \mathcal{Ap}(\Delta)$  and  $x \in \text{Ch}(A)$  we have*

$$\begin{aligned}\mathcal{M}(A) &= \{M \cap A : M \in \mathcal{M}(\Sigma) \text{ and } M \cap A \neq \emptyset\}, \\ &= \{\pi_{A,x}(M) : M \in \mathcal{M}(\Sigma) \text{ and } M \cap A \neq \emptyset\}, \\ &= \{\pi_{A,x}(M) : M \in \mathcal{M}(\Sigma)\}.\end{aligned}$$

*ii) For  $A \in \mathcal{Ap}(\Delta)$  and  $x \in \text{Ch}(A)$  we have*

$$\mathcal{M}(\Sigma) = \bigcup_{m \in \mathcal{M}(A)} C(\pi_{A,x}^{-1}(m)),$$

where  $C(\pi_{A,x}^{-1}(m))$  denotes the set of all the connected components of  $\pi_{A,x}^{-1}(m)$ .

Building-walls (resp. walls) bound dials of buildings (resp. half-spaces) so a similar fact holds for dials of building and half-spaces.

We use the following terminology to describe a building-wall relatively to some chambers.

**Definition 5.22.** *Let  $M \in \mathcal{M}(\Sigma)$  and  $E, F \subset \Sigma$ .*

- i) We say that  $M$  crosses  $E$  if  $E \setminus M$  has at least two connected components.*
- ii) We say that the building-wall  $M$  separates  $E$  and  $F$  if the interior of  $E$  and  $F$  are entirely contained in two distinct connected components of  $\Sigma \setminus M$ .*

The metric over the chambers is determined by the building-wall structure.

**Proposition 5.23** ([Cla13, Proposition 5.8]). *Let  $x_1$  and  $x_2$  be two chambers. If we denote  $d_c(\cdot, \cdot)$  the metric on the chamber system, then*

$$d_c(x_1, x_2) = \#\{M \in \mathcal{M}(\Sigma) : M \text{ separates } x_1 \text{ and } x_2\}.$$

In a right-angled building it appears that two distinct building-walls are either orthogonal or do not intersect. This explains the following terminology and notation.

**Notation.** *Let  $M$  and  $M'$  be two distinct building-walls.*

i) if  $M \cap M' \neq \emptyset$  we write  $M \perp M'$  and we say that  $M$  is orthogonal to  $M'$ ,

ii) if  $M \cap M' = \emptyset$  we write  $M \parallel M'$  and we say that  $M$  is parallel to  $M'$ .

Clearly, if  $D, D' \in \mathcal{D}(\Sigma)$  are bounded by  $M, M' \in \mathcal{M}(\Sigma)$  with if  $M \perp M'$ , then  $D \cap D'$  contains a chamber. On the other hand, if  $M \parallel M'$  then there exists  $D$  bounded by  $M$  and  $D'$  bounded  $M'$  such that  $M' \subset D$  and  $M \subset D'$ .

## 5.8 Geometric characterization of parabolic subgroups

Now we discuss residues in  $\Sigma$ , parallel to the discussion at the end of Subsection 5.2.

**Notation.** For  $I \subset S$  and  $g \in \Gamma$ , let  $g\Sigma_I$  denote the union of the chambers of the  $I$ -residue containing  $gx_0$ .

Notice that  $g\Sigma_I = g\Gamma_I x_0$  and  $\text{Ch}(g\Sigma_I) = g\Gamma_I$ . For simplicity, in the following we also call a subset  $g\Sigma_I \subset \Sigma$  a *residue*. Notice that a rotation around a building-wall that crosses  $g\Sigma_I$  is of the form  $g\gamma s^\alpha \gamma^{-1} g^{-1}$  with  $s \in I$ ,  $s^\alpha \neq e$  and  $\gamma \in \Gamma_I$ . By the definitions of the action and the residues we obtain the following fact.

**Fact 5.24.** Let  $R = g\Sigma_I$  be a residue. Then

- $R$  is stabilized by the rotations around the building-walls that cross it,
- $\text{Stab}_\Gamma(R) = g\Gamma_I g^{-1}$  is generated by these rotations,
- The type  $I$  of  $R$  is equal to the set of types of all the building-walls  $M$  satisfying

$$M \text{ crosses } R \text{ and } M \text{ is along } gx_0.$$

The following result gives a converse to the preceding fact. We recall that a set of chambers is convex if it is convex for the combinatorial metric over the chambers (see Definition 5.3).

**Theorem 5.25.** Let  $C \subset \text{Ch}(\Sigma)$  be a convex set of chambers. Let  $R = \bigcup_{x \in C} x \subset \Sigma$  and let  $P_R$  denote the group generated by the rotations around the building-walls that cross  $R$ . If  $P_R$  stabilizes  $R$ , then  $R$  is a residue in  $\Sigma$ .

*Proof.* Up to a translation on  $R$  and a conjugation on  $P_R$ , we can assume that  $x_0 \subset R$ . We start by proving that  $P_R$  acts freely and transitively on the set of chambers  $C$ . The action of  $\Gamma$  is free thus the action of  $P_R$  is free. For  $x \in C$ , by convexity of  $C$ , there exists a gallery

$$x_0 \sim x_1 \sim \cdots \sim x_\ell = x$$

of distinct chambers in  $C$ . Let  $M_i$  be the building-wall of type  $s_i \in S$  between  $x_{i-1}$  and  $x_i$ . Then

$$s_1^{\alpha_1} x_0 = x_1, s_1^{\alpha_1} s_2^{\alpha_2} x_0 = x_2, \dots, s_1^{\alpha_1} \dots s_\ell^{\alpha_\ell} x_0 = x,$$

for some exponent  $\alpha_i \in \mathbb{Z}$ .

We notice that  $s_1^{\alpha_1} \dots s_{i-1}^{\alpha_{i-1}} s_i (s_1^{\alpha_1} \dots s_{i-1}^{\alpha_{i-1}})^{-1}$  is a rotation around  $M_i$ . Therefore  $x$  may be obtained from  $x_0$  by successive rotations around the building-walls  $M_i$ . These building-walls cross  $R$ , thus the action is transitive. This proves that  $R = P_R x_0$  and  $\text{Stab}_\Gamma(R) = P_R$ . It remains to prove that  $P_R$  is of the form  $\Gamma_I$  for a certain  $I \subset S$ .

We set  $I \subset S$  the set of all the types of the building-walls that cross  $R$  along  $x_0$  and we identify  $P_R$  with  $\Gamma_I$ . The inclusion  $\Gamma_I < P_R$  comes from the definitions of  $P_R$  and  $I$ . We proceed by induction on  $d_c(x_0, gx_0) = \ell$  to check that every element  $g$  of  $P_R$  is a product of elements of  $\Gamma_I$ . If  $\ell = 0$ , there is nothing to say. If  $\ell > 0$  we choose  $g = s_1^{\alpha_1} \dots s_\ell^{\alpha_\ell}$  such that  $d_c(x_0, gx_0) = \ell$ . By convexity,  $s_1^{\alpha_1} x_0 \in C$  so  $s_1 \in \Gamma_I$ . Indeed  $s_1$  is a rotation around a building-wall that crosses  $R$  along  $x_0$ . Then  $d_c(x_0, s_1^{-\alpha_1} g x_0) = \ell - 1$  and  $s_1^{-\alpha_1} g \in P_R$ . The induction assumption allows us to conclude.  $\square$

In particular, this last theorem is used in Subsection 6.1 when we discuss the boundaries of the residues in the hyperbolic case. Finally we recall that the intersections of parabolic subgroups in  $\Gamma$  (resp. in  $W$ ) is again a parabolic subgroup.

## 5.9 $\Sigma$ as a metric space

A natural geodesic metric on  $\Sigma$  is obtained as follows. We designate by  $D$  the Davis chamber of  $\Gamma$ . We recall that  $D$  is obtained from  $D^{(1)}$  by attaching a  $k$ -cube to every subgraph that is isomorphic to the 1-skeleton of a  $k$ -cube. Now, for any  $k$ , we equip each  $k$ -cube of  $D$  with the Euclidean metric of the  $[0, 1]^k$ .

The polyhedral metric  $d(\cdot, \cdot)$  induced on  $\Sigma$  by this construction is geodesic and complete. Moreover, any automorphism of  $\Delta$  is an isometry of  $(\Sigma, d)$ . In particular,  $\Gamma$  acts geometrically on  $(\Sigma, d)$  and, as  $\Gamma$  is assumed to be hyperbolic,  $(\Sigma, d)$  is a hyperbolic metric space.

In  $(\Sigma, d)$  the building-walls are convex and connected subsets and we can precise this descriptions using geodesic rays. Let  $M \in \mathcal{M}(\Sigma)$  be of type  $s$ , let  $x \in \text{Ch}(\Sigma)$  such that  $M$  is along  $x$  and let  $\sigma_s$  be the maximal face of type  $s$  of  $x$ . We denote by  $\text{Ext}(\sigma_s)$  the set of all the geodesic rays such that there exists  $\epsilon > 0$  with  $r([0, \epsilon)) \subset \sigma_s$ . Then  $M = \{r(t) : r \in \text{Ext}(\sigma_s) \text{ and } t \in [0, +\infty)\}$ .

However in the case when  $W$  is a reflection group of the hyperbolic space  $\mathbb{H}^d$  it seems more natural to equip  $D$  with the hyperbolic metric. Then  $D$  is isometric to the Coxeter polytope provided by  $W$ . We designate by  $d'(\cdot, \cdot)$  the piecewise hyperbolic metric on  $\Sigma$  induced by this construction. This metric satisfies the same properties stated above (geodesic, complete, hyperbolic and admits a geometric action of  $\Gamma$ ). The two metrics  $(\Sigma, d)$  and  $(\Sigma, d')$  are quasi-isometric. Since our goal is to study  $\partial\Gamma$ , it makes no difference to consider  $(\Sigma, d)$  or  $(\Sigma, d')$ . However, the arguments presented in Sections 6, 7, 8, and 9 hold in the generic case so we consider  $(\Sigma, d)$  in those sections. For Section 10, which focuses on hyperbolic reflection groups, it will be more convenient to consider  $(\Sigma, d')$ .

### 5.10 Boundary of the building

In this subsection we describe some basic properties of  $\partial\Gamma$ . In the sequel, we use the geometric action of  $\Gamma$  on  $(\Sigma, d)$  to identify  $\partial\Gamma$  and  $\partial\Sigma$ . Now consider a building-wall  $M$  of type  $s$ . We have  $\partial M \simeq \partial\text{Stab}_\Gamma(M)$  using the previous identification. Consider a subgroup  $P < \Gamma$ , as  $\Gamma$  is hyperbolic, either  $\partial P = \partial\Gamma$ , or  $\partial P$  is of empty interior. Hence, here there are two possible cases:

- $\partial M \simeq \partial\Gamma$ ,
- $\text{Int}(\partial M)$  is empty.

Moreover, the hyperbolicity assumption implies the following lemma.

**Lemma 5.26.** *Let  $M$  and  $M'$  be two distinct building-walls. If  $M \parallel M'$  then  $\partial M \cap \partial M' = \emptyset$ .*

*Proof.* Let  $\gamma : [0, +\infty) \rightarrow \Sigma$  be a geodesic ray contained in  $M$ . For simplicity we denote by  $\gamma$  the image of  $\gamma$ . Assume that there exists  $K > 0$  such that  $\text{dist}(\gamma(t), M') \leq K$  for every  $t \geq 0$ . Since  $M \cap M' = \emptyset$  and because of the chamber structure, there exists  $K' > 0$  such that  $K' \leq \text{dist}(\gamma(t), M')$  for every  $t \geq 0$ .

Now let  $\Gamma'$  be the group generated by a rotation  $r$  around  $M$  and a rotation  $r'$  around  $M'$ . If the rotations are of order two, then  $\Gamma'$  is an infinite dihedral group and the subset  $\Gamma'\gamma \subset \Sigma$  is quasi-isometric to a Euclidean half space. If the rotations are of order larger than two then  $\Gamma'\gamma$  contains a proper subset quasi-isometric to a Euclidean half space.  $\square$

Now we can precise the description of the two cases above. In the first case,  $s$  commutes with every generator  $r \in S$ . In the Davis complex this means that all the other building-walls are orthogonal to  $M$ . Then  $\text{Stab}_\Gamma(M) = \Gamma$ .

In the second case, there exists  $r \in S$  that does not commute with  $s$ . In the Davis complex this means that there exists a building-wall  $M'$  parallel to  $M$ . This implies that  $\partial M \subsetneq \partial\Gamma$ . In this case,  $\partial\Gamma \setminus \partial M$  is the disjoint union  $\text{Int}(\partial D_0(M)) \sqcup \dots \sqcup \text{Int}(\partial D_{q-1}(M))$  where  $D_0(M), \dots, D_{q-1}(M)$  are the dials of building bounded by  $M$ . Naturally a rotation around  $M$  extends to the boundary as an homeomorphism that permutes  $\partial D_0(M), \dots, \partial D_{q-1}(M)$  and fixes  $\partial M$ . Moreover  $\text{Int}(\partial M) = \emptyset$ ,  $\text{Int}(\partial D_i(M)) \neq \emptyset$ , and the topological boundary of  $\partial D_i(M)$  in  $\partial\Gamma$  is  $\partial M$  for any  $i = 0, \dots, q-1$ . Concerning the dials of building, the following alternative holds. Let  $D$  be a dial of building bounded by the building-wall  $M$ , then:

- either  $\partial M = \partial D = \partial\Gamma$ ,
- or the topological boundary of  $\partial D$  in  $\partial\Gamma$  is  $\partial M$ . In this case,  $\text{Int}(\partial D(M)) \neq \emptyset$  and  $\text{Int}(\partial M) = \emptyset$ .

In [BK13, Proposition 5.2.], it is proved that the boundaries of half-spaces in a hyperbolic Coxeter group form a basis for the visual topology on the visual boundary. In the case of right-angled building the analogous statement holds.



**Fact 5.27.** *The topology generated by  $\{\partial D : D \in \mathcal{D}(\Sigma)\}$  is equivalent to the topology induced by a visual metric on  $\partial\Gamma$ .*

Finally, consider an apartment  $A$  containing the base chamber  $x_0$  and the retraction map  $\pi_{A,x_0} : \Sigma \rightarrow A$ . This retraction maps any geodesic ray of  $\Sigma$  starting from a based point  $p_0 \in x_0$  to a geodesic ray in  $A$  starting from  $p_0$ . Hence  $\pi_{A,x_0}$  extends naturally to the boundaries and we keep the notation  $\pi_{A,x_0} : \partial\Sigma \rightarrow \partial A$  for this extension.

**Remark 5.28.** *In [DM02], M.W. Davis and J. Meier described how properties of connectedness of  $\partial\Gamma$  are encoded in the combinatorial structure of  $\mathcal{G}$ . We use a corollary of their result in Subsection 6.2.*

**Remark 5.29.** *A classification of F. Haglund and F. Paulin states that the construction presented in Subsection 5.5 describes all the right-angled buildings in the following sense.*

**Theorem 5.30** ([HP03, Proposition 5.1.]). *Let  $\Gamma$  be the graph product given by the pair  $(\mathcal{G}, \{G_i\}_{i=1,\dots,n})$  as in Definition 5.13. Let  $\Delta$  be the building of type  $(W, S)$  associated with  $\Gamma$ . Assume that  $\Delta'$  is a building of type  $(W, S)$  such that for any  $s_i \in S$  the  $\{s_i\}$ -residues of  $\Delta'$  are of cardinality  $\#G_i$ . Then  $\Delta$  and  $\Delta'$  are isomorphic.*

## 6 Curves in connected parabolic limit sets

As we will see with Example 6.8, parabolic limit sets (*i.e.* boundaries of residues of the building) play a key role in the proof of the CLP on boundaries of graph product.

In this section, we use the ideas of [BK13, Section 5 and 6] to prove Theorem 6.12 which is the first major step to prove the main result of this article (Theorem 10.1). The idea of this theorem is to control the modulus of the curves of a parabolic limit set by the modulus of curves in the neighborhood of a single curve. Then we apply this theorem to recover a result about boundaries of right-angled Fuchsian buildings.

We use the notation and convention of Section 5. In particular  $\Gamma$  is a fixed graph product given by the pair  $(\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$ . We identify the building  $\Delta$  with its Davis complex  $\Sigma$  equipped with the piecewise Euclidean metric. The base chamber is  $x_0$ . We assume that  $\Gamma$  and  $\Sigma$  are hyperbolic and that  $\partial\Gamma$  is arcwise connected. The metric on  $\text{Ch}(\Sigma)$  is denoted by  $d_c(\cdot, \cdot)$ . Moreover, in this section we equip  $\partial\Gamma$  with a self-similar metric that comes from the action of  $\Gamma$  on  $\Sigma$ .

### 6.1 Parabolic limit sets in $\partial\Gamma$

In this subsection we give some basic properties of boundaries of parabolic subgroups. At the end of this subsection we will see that these subsets of the boundary cause difficulties in proving the CLP.

**Definition 6.1.** Let  $P = g\Gamma_I g^{-1}$  be a parabolic subgroup of  $\Gamma$ . If the limit set of  $P$  in  $\partial\Gamma$  is non-empty, we call it a parabolic limit set. If moreover  $\partial P \neq \partial\Gamma$  the parabolic limit set is called a proper parabolic limit set.

Equivalently we could say that a subset  $F \subset \partial\Gamma$  is a parabolic limit set if there exists a residue  $g\Sigma_I$  such that  $F$  is equal to  $\partial(g\Sigma_I)$  under the canonical homeomorphism between  $\partial\Gamma$  and  $\partial\Sigma$ . In the following we will frequently use this point of view about parabolic limit sets.

The following notion of convex hull of a subset of the boundary will be used in this section.

**Definition 6.2.** Let  $F$  be a subset of  $\partial\Gamma$  containing more than one point and such that  $\overline{F} \neq \partial\Gamma$ . Let

$$\mathcal{D}^c(F) = \{D \in \mathcal{D}(\Sigma) : F \subset \partial\Gamma \setminus \partial D\}.$$

The convex hull of  $F$  in  $\Sigma$  is defined by

$$\text{Conv}(F) = \Sigma \setminus \bigcup_{D \in \mathcal{D}^c(F)} D.$$

If  $\overline{F} = \partial\Gamma$  then we set  $\text{Conv}(F) = \Sigma$ .

Clearly we can also write  $\text{Conv}(F) = \bigcap_{D \in \mathcal{D}^c(F)} \Sigma \setminus D$ . Hence  $\text{Conv}(F)$  is convex for both the geodesic metric on  $\Sigma$  and the combinatorial metric over the chambers of  $\Sigma$ . Moreover we observe that  $F \subset \partial\text{Conv}(F)$ .

**Example 6.3.** Let  $\partial P$  be a parabolic limit set and assume that  $P = \Gamma_I$ . Then we can verify that  $\text{Conv}(\partial P) = \Sigma_J$  where  $J = I \cup \{s_j \in S : s_j s_i = s_i s_j \text{ for any } s_i \in I\}$ .

In particular, if  $M \in \mathcal{M}(\Sigma)$  then  $\text{Conv}(\partial M)$  is the union of all the chambers along  $M$ .

In the following definition  $\overline{\Sigma} = \Sigma \cup \partial\Sigma$  and if  $M$  is a building-wall  $\overline{M} = M \cup \partial M$ .

**Definition 6.4.**

- i) Let  $F$  be a subset of  $\partial\Sigma$ . We say that a building-wall  $M$  cuts  $F$  if there exist two distinct indices  $i$  and  $j$  such that  $F$  meets both  $\partial D_i(M)$  and  $\partial D_j(M)$ .
- ii) If  $E_1 \subset \partial\Sigma$  and  $E_2 \subset \Sigma$  (resp.  $E_2 \subset \partial\Sigma$ ) we say that a building-wall  $M$  separates  $E_1$  and  $E_2$  if  $E_1$  and  $E_2$  are entirely contained in two distinct connected components of  $\overline{\Sigma} \setminus \overline{M}$ .

The proof of the following fact is identical to the proof of [BK13, Lemma 5.7].

**Fact 6.5.** Let  $F$  be a subset of  $\partial\Sigma$ . The building-wall  $M$  cuts  $F$  if and only if  $M$  crosses  $\text{Conv}(F)$  (see Definition 5.22).

The following corollary is an immediate consequence of the preceding fact and of Theorem 5.25

**Corollary 6.6.** *Let  $F$  be a subset of  $\partial\Sigma$  containing at least two distinct points and  $P_F$  denote the group generated by the rotations around the building-walls that cut  $F$ . If  $P_F$  stabilizes  $F$ , then  $F$  is a parabolic limit set.*

This characterization yields the following corollary concerning the connectedness of the parabolic limit sets.

**Corollary 6.7.** *Let  $\partial P$  be a parabolic limit set. Then every connected component  $F$  of  $\partial P$  containing more than one point is a parabolic limit set.*

*Proof.* Let  $M$  be a building-wall that cuts  $F$ . Since  $M$  cuts  $\partial P$  a rotation  $r \in \Gamma$  around  $M$  stabilize  $\partial P$  so in particular it permutes the connected components of  $\partial P$ . With  $r(\partial M \cap F) = \partial M \cap F$  we deduce that  $r(F) = F$  and so  $F$  is a parabolic limit set.  $\square$

Finally the following example illustrates the difficulty caused by parabolic limit sets in proving the CLP.

**Example 6.8.** *Let  $M \in \mathcal{M}(\Sigma)$  be a building-wall of type  $s$  along the base chamber  $x_0$ . Let  $P = \text{Stab}_\Gamma(M)$ . The group  $P$  is the parabolic subgroup which is generated by the generators  $r \in S \setminus \{s\}$  such that  $rs = sr$ . Moreover, as we recalled in Subsection 5.10,  $\partial P \simeq \partial M$ . Now we assume that  $\partial P$  is a proper parabolic limit set and we pick  $g \in \Gamma$ . Now we verify that*

- *either  $g\partial P = \partial P$ ,*
- *or  $\partial P \cap g\partial P = \emptyset$ .*

*Indeed if two building-walls  $M_1$  and  $M_2$  are distinct with  $M_1 \perp M_2$  then  $M_1$  and  $M_2$  are of distinct types. As  $M$  and  $gM$  are of the same type it follows that if  $M \cap gM \neq \emptyset$  then  $M = gM$  and  $g\partial P = \partial P$ . On the other hand, if  $M \cap gM = \emptyset$ , by Lemma 5.26 the hyperbolicity implies that  $\partial P \cap g\partial P = \emptyset$ .*

*Finally the set  $\cup_{g \in \Gamma} g\partial M$  is the union of countably many disjoint copies of  $\partial M$ . In the introduction we recalled that an efficient way to prove the CLP is to follow curves using bi-Lipschitz maps (see Section 4). As  $\Gamma$  acts by bi-Lipschitz homeomorphisms on its boundary, the first idea is to use  $\Gamma$  to follow curves. However if a non-constant curve  $\eta$  is contained in  $\partial M$  then we cannot hope to follow the curves of  $\partial\Gamma$  using  $\eta$  and  $\Gamma$ .*



Figure 2: Example 6.8 on the boundary of a thin building

## 6.2 Modulus of curves in connected parabolic limit set

In this subsection we apply the ideas of [BK13, Section 5 and 6] to  $\Gamma$ . As in Subsection 3.2,  $d_0$  denotes a small constant compared with  $\text{diam } \partial\Gamma$  and with the constant of approximate self-similarity. Then  $\mathcal{F}_0$  is the set of curves of diameter larger than  $d_0$ . Here we prove that, from the point of view of the modulus, curves in a parabolic limit set are all the same (see consequences of Theorem 6.12).

Until the end of this article, we use the following notation:

**Notation.** Let  $\partial P$  be a connected parabolic limit set in  $\partial\Gamma$ . For  $\delta, r > 0$ , let  $\mathcal{F}_{\delta,r}(\partial P)$  denote the set of curves  $\gamma$  in  $\partial\Gamma$  such that:

- $\text{diam } \gamma \geq d_0$ ,
- $\gamma \subset N_\delta(\partial P)$ ,
- $\gamma \not\subset N_r(\partial Q)$  for any connected parabolic limit set  $\partial Q \subsetneq \partial P$ .

As we saw in Example 6.8, it is impossible to use the curves in the parabolic limit set to follow other curves. Nevertheless, in this section we prove that these curves can be used to follow the curves in  $\partial P$  (Proposition 6.11). Then we deduce from this proposition a control of the modulus of the curves in parabolic limit sets (Theorem 6.12). To this end we use the following notion.

**Definition 6.9.** Let  $L \geq 0$  and  $I$  a non-empty subset of  $S$ . A curve  $\gamma$  in  $\partial\Gamma$  is called a  $(L, I)$ -curve if

- $x_0 \subset \text{Conv}(\gamma)$ ,
- for all  $s \in I$ , there exists a panel  $\sigma_s$  of type  $s$  inside  $\text{Conv}(\gamma)$  with  $\text{dist}(x_0, \sigma_s) \leq L$ .

The next proposition says that curves in parabolic limit sets are  $(L, I)$ -curves.

**Proposition 6.10.** *Let  $I \subset S$  and  $P = h\Gamma_I h^{-1}$ . Then for all  $r > 0$ , there exist  $L > 0$  and  $\delta > 0$  such that if  $x_0 \subset \text{Conv}(\gamma)$  and  $\gamma \in \mathcal{F}_{\delta, r}(\partial P)$ , then  $\gamma$  is a  $(L, I)$ -curve.*

*Proof.* We fix  $r > 0$  and we assume that for every integer  $n \geq 1$ , there exists a curve  $\gamma_n$  such that:

- $x_0 \subset \text{Conv}(\gamma_n)$ ,
- $\gamma_n \in \mathcal{F}_{1/n, r}(\partial P)$ ,
- $\gamma_n$  is not a  $(n, I)$ -curve.

For  $n \geq 1$ , we designate the ball of center  $x_0$  and of radius  $n$  for the metric over the chambers by

$$B_c(x_0, n) = \{x \in \text{Ch}(\Sigma) : d_c(x_0, x) \leq n\}.$$

For simplicity we also designate by  $B_c(x_0, n)$  the union of its chambers. Up to a subsequence we can suppose that for a fixed  $s \in I$ , there is no panel of type  $s$  in  $B_c(x_0, n) \cap \text{Conv}(\gamma_n)$  for  $n \geq 1$ .

We want to reveal a contradiction using the sequences  $\{\gamma_n\}_{n \geq 1}$  and  $\{\text{Conv}(\gamma_n)\}_{n \geq 1}$ . According to [Mun75, p. 281] the set of non-degenerate continua in a compact metric space is a compact metric space with respect to the Hausdorff distance. Therefore, up to a subsequence, we can suppose that  $\gamma_n$  tends to a non-degenerate continuum  $\mathcal{L} \subset \partial P$ .

Since  $x_0 \subset \text{Conv}(\gamma_n)$ , using a diagonal argument we can also assume that, up to a subsequence,  $\text{Conv}(\gamma_{n+k}) \cap B_c(x_0, n)$  is non-empty and constant for  $k \geq 0$ . We denote  $C := \bigcup_{n \geq 1} \text{Conv}(\gamma_n) \cap B_c(x_0, n)$ . As we remarked before, if  $F \subset \partial \Gamma$  contains more than one point  $F \subset \partial \text{Conv}(F)$ . In particular  $\mathcal{L} \subset \partial C$ .

Now, by the assumptions on the sequence  $\{\gamma_n\}_{n \geq 1}$  we observe that  $C$  does not contain any panel of type  $s$ . Hence  $\mathcal{L}$  is contained in the limit set of a parabolic subgroup of the form  $g\Gamma_J g^{-1}$  with  $s \notin J$ . Moreover, we know that intersections of parabolic subgroups are parabolic subgroups (see [Cla13, Theorem 1.2]). Therefore  $\mathcal{L}$  is contained in the limit set of the parabolic subgroup  $Q' = P \cap g\Gamma_J g^{-1}$ . Let  $\partial Q$  be the connected component of  $\partial Q'$  that contains  $\mathcal{L}$ . Thanks to Corollary 6.7,  $\partial Q$  is a parabolic limit set. Now, as  $\gamma_n$  tends to  $\mathcal{L} \subset \partial Q$  with respect to the Hausdorff distance, we have that for  $n \geq 0$  large enough  $\gamma_n \subset N_r(\partial Q)$ .

Finally, if  $\partial Q \subsetneq \partial P$ , this reveals a contradiction with  $\gamma_n \in \mathcal{F}_{1/n, r}(\partial P)$  for every  $n \geq 1$ . If  $\partial Q = \partial P$ , then we apply the same reasoning with  $s' \in I \setminus \{s\}$ .

□

An interesting feature of  $(L, I)$ -curves is that these curves are crossed by building-walls of type in  $I$ . Which means that from a  $(L, I)$ -curve, we can follow curves using rotations around building-walls of type in  $I$ .

**Proposition 6.11.** *Let  $\epsilon > 0$ ,  $L > 0$  and  $I$  be a non-empty subset of  $S$ . For  $P = h\Gamma_I h^{-1}$ , let  $\eta$  denote a curve contained in  $\partial P$ . Then there exists a finite subset  $F \subset \Gamma$  such that for any  $(L, I)$ -curve  $\gamma$  the set  $\bigcup_{g \in F} g\gamma$  contains a curve that belongs to  $\mathcal{U}_\epsilon(\eta)$ .*

*Proof.* We divide the proof in four steps. In this proof  $M_s$  denotes the building-wall of type  $s \in S$  along  $x_0$ .

i) First, we can suppose without loss of generality that  $P = \Gamma_I$ . Indeed, as  $h \in \Gamma$  is a bi-Lipschitz homeomorphism of  $(\partial\Gamma, d)$ , then if the property holds for  $\Gamma_I$  it holds for  $h\Gamma_I h^{-1}$ .

ii) Now we prove that the following property holds.

*The set  $\bigcup_{g \in F_L} g\gamma$  contains a curve passing through  $\partial M_s$  for every  $s \in I$ ,*

where  $F_L = \{g \in \Gamma : |g| \leq L\}$  and  $|g| = d_c(x_0, gx_0)$ .

As  $\gamma$  is a  $(L, I)$ -curve, for any  $s \in I$  there exist  $\alpha \in \mathbb{Z}$  and  $g \in F_L$  such that  $gx_0$  and  $gs^\alpha x_0$  belong to  $\text{Conv}(\gamma)$ . We fix  $s \in I$ ,  $\alpha \in \mathbb{Z}$  and  $g \in F_L$  as before and let  $x_0 \sim g_1 x_0 \sim \dots \sim g_\ell x_0$  be a gallery contained in  $\text{Conv}(\gamma)$  with  $g_{\ell-1} = gs^\alpha$  and  $g_\ell = g$ . For any  $i = 0, \dots, \ell-1$ , let  $M_i$  denote the building-wall separating  $g_i x_0$  and  $g_{i+1} x_0$ . In particular,  $\partial M_i$  cuts  $\gamma$  for any  $i = 0, \dots, \ell-1$ . This means that if  $M_i$  is of type  $s_i$ , then  $\gamma \cap g_i s_i^{\alpha_i} g_i^{-1} \gamma \neq \emptyset$  for any  $\alpha_i \in \mathbb{Z}$ . In particular, if  $\alpha_i$  is such that  $g_{i+1} = g_i s_i^{-\alpha_i}$  then  $\gamma \cap g_i g_{i+1}^{-1} \gamma \neq \emptyset$  for any  $i = 0, \dots, \ell-1$ . Hence the set  $\gamma \cup g_1^{-1} \gamma \cup \dots \cup g_\ell^{-1} \gamma$  is arcwise connected and  $g_\ell^{-1} \gamma$  intersects  $\partial M_s$ . Thus the property is satisfied.

iii) Let  $\Sigma_I \subset \Sigma$  be the residue associated with  $\Gamma_I$ . We recall that this means  $\Sigma_I = \Gamma_I x_0$ . For each  $1 \leq i \leq k$  let  $D_i$  be a dial of building bounded by the building-wall  $M_i$ . We assume that each  $D_i$  intersects  $\Sigma_I$  properly (i.e.  $\Sigma_I \cap D_i \neq \emptyset$  and  $\Sigma_I \cap D_i \neq \Sigma_I$ ). In particular, this means that the building-walls  $M_1, \dots, M_k$  have their types contained in  $I$ .

Now we prove that the following property holds.

*There exists a finite subset  $F_0 \subset \Gamma$  such that for every  $(L, I)$ -curve  $\gamma$  the set  $\bigcup_{g \in F_0} g\gamma$  contains a curve passing through  $\partial D_1, \dots, \partial D_k$ .*

For  $i = 1, \dots, k$  pick  $h_i \in \Gamma_I$  such that  $M_i$  is along  $h_i x_0 \in \Sigma_I$ . In particular, for any  $i$ , we can write  $M_i = h_i(M_s)$  where  $s \in I$  is the type of  $M_i$ . Let  $g_1 x_0 = h_1 x_0 \sim g_2 x_0 \sim \dots \sim g_\ell x_0 = h_k x_0$  be a gallery in  $\Sigma_I$  passing through  $h_1 x_0, \dots, h_k x_0$  in this order.

Applying the second step of the proof, there exists a curve  $\theta$  in  $\bigcup_{g \in F_L} g\gamma$  such that  $\theta$  crosses  $\partial M_s$  for every  $s \in I$ . Therefore the set  $\bigcup_{i=1, \dots, \ell} g_i \theta$  meets  $g_i(M_s)$  for any  $i = 1, \dots, k$  and any  $s \in I$ . In particular, it meets every  $h_i(M_s)$  and intersects every  $\partial D_1, \dots, \partial D_k$ .

We set  $F_0 = \{g_i g \in \Gamma : |g| \leq L, 1 \leq i \leq \ell\}$ , and it is now enough to check that  $\bigcup_{g \in F_0} g\gamma$  is arcwise connected. For any  $i = 1, \dots, \ell-1$  let  $s_i \in I$  and  $\alpha_i \in \mathbb{Z}$  be such that  $g_{i+1} = g_i s_i^{\alpha_i}$ . Then  $g_{i+1} \theta = (g_i s_i^{\alpha_i} g_i^{-1}) g_i \theta$ . Since  $\theta$  intersects any  $\partial M_{s_i}$  then  $g_i \theta \cap g_i(\partial M_{s_i}) \neq \emptyset$  and this intersection is fixed by  $g_i s_i^{\alpha_i} g_i^{-1}$ . Thus  $g_i \theta \cap g_{i+1} \theta \neq \emptyset$ .

iv) With Fact 5.27, we can choose,  $D'_1, \dots, D'_{k+1}$  a collection of dials of building such that the union of their boundaries is a neighborhood of  $\eta$  contained in the  $\epsilon/2$  neighborhood

of  $\eta$ . We also assume that  $\eta$  enters in the boundaries of the  $D'_1, \dots, D'_{k+1}$  in this order. For any  $i = 1, \dots, k+1$ , let  $r_i$  denote the rotation around the building-wall associated with  $D'_i$ . Let  $D_1, \dots, D_k$  be a collection of dials of building such that  $\partial D_i \subset \partial D'_i \cap \partial D'_{i+1}$ . Applying the previous step of the proof, there exists a finite set  $F_0 \subset \Gamma$  such that for every  $(L, I)$ -curve  $\gamma$  the set  $\bigcup_{g \in F_0} g\gamma$  contains a curve passing through each  $\partial D_1, \dots, \partial D_k$ .

If for some  $i = 1, \dots, k+1$  the curve  $\eta$  leaves  $\partial D'_i$  then  $\theta \bigcup_{\alpha \in \mathbb{Z}} r_i^\alpha \theta$  contains a curve that does not leave  $\partial D'_i$ . Finally we set  $F = \{r_i^\alpha g : \alpha \in \mathbb{Z} \text{ and } g \in F_0\}$  and  $F$  satisfies the desired property.  $\square$

We use the two preceding propositions to obtain a control of  $\text{Mod}_p(\mathcal{F}_{\delta,r}(\partial P))$ .

**Theorem 6.12.** *There exists an increasing function  $\delta_0 : (0, +\infty) \rightarrow (0, +\infty)$  satisfying the following property. Let  $p \geq 1$ , let  $\eta \in \mathcal{F}_0$ , and let  $\partial P$  be the smallest parabolic limit set containing  $\eta$ . Let  $r > 0$  be small enough so that  $\eta \notin \overline{N_r}(\partial Q)$  for any connected parabolic limit set  $\partial Q \subsetneq \partial P$  and let  $\delta < \delta_0(r)$ . Then for  $\epsilon > 0$  small enough there exists a constant  $C = C(d_0, p, \eta, r, \epsilon)$  such that for every  $k \geq 1$*

$$C^{-1} \cdot \text{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k) \leq \text{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G_k) \leq C \cdot \text{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

Furthermore when  $p$  belongs to a compact subset of  $[1, +\infty)$  the constant  $C$  may be chosen independent of  $p$ .

*Proof.* i) We can assume, without loss of generality, that  $x_0 \subset \text{Conv}(\gamma)$  for every  $\gamma \in \mathcal{F}_0$ . Indeed, there exists an upper bound  $N$  depending on  $d_0$  such that  $\text{dist}(x_0, \text{Conv}(\gamma)) \leq N$  for every  $\gamma \in \mathcal{F}_0$ . So there exists only a finite set  $E$  of elements of  $\Gamma$  such that for  $g \in E$ , there exists  $\gamma \in \mathcal{F}_0$  with  $\text{dist}(x_0, \text{Conv}(\gamma)) = d_c(x_0, gx_0)$ . Now, for  $\mathcal{F} \subset \mathcal{F}_0$ , we write

$$\mathcal{F}^{x_0} = \{\gamma \in \mathcal{F} : x_0 \subset \text{Conv}(\gamma)\}.$$

Applying Proposition 3.11  $N$  times and Proposition 2.3, we obtain that for every  $k \geq 1$  large enough

$$\text{Mod}_p(\mathcal{F}^{x_0}, G_k) \leq \text{Mod}_p(\mathcal{F}, G_k) \leq C \cdot \text{Mod}_p(\mathcal{F}^{x_0}, G_k),$$

where  $C$  depends only on  $d_0$ .

ii) Similarly, for  $\epsilon > 0$  small enough, we can assume that  $\mathcal{U}_\epsilon(\eta) \subset \mathcal{F}_0$ . Indeed, for  $\epsilon > 0$  small enough there exists an upper bound  $N$  depending on  $d_0$  such that if  $\gamma \in \mathcal{U}_\epsilon(\eta)$  then  $\text{dist}(x_0, \text{Conv}(\gamma)) \leq N$ . Again, the multiplicative constant induced by this assumption depends only on  $d_0$ .

iii) As a consequence of the preceding part, for  $\epsilon > 0$  small enough, one has  $\mathcal{U}_\epsilon(\eta) \subset \mathcal{F}_{\delta,r}(\partial P)$  and the left-hand side inequality is established by Proposition 2.3 (1).

iv) Now we prove the right-hand side inequality. Let  $P = h\Gamma_I h^{-1}$ , let  $\eta$  be a curve in  $\partial P$ ,  $r > 0$  as in the hypothesis of the theorem and  $\epsilon > 0$  small enough so that the preceding part hold.

With the assumption of the first part, we can apply Proposition 6.10 and set  $L > 0$  and  $\delta > 0$  such that the curves of  $\mathcal{F}_{\delta,r}(\partial P)$  are  $(L, I)$ -curves. Let  $F \subset \Gamma$  be the finite set given by Proposition 6.11 and let  $\rho : G_k \rightarrow [0, +\infty)$  be a  $\mathcal{U}_\epsilon(\eta)$ -admissible function. We define  $\rho' : G_k \rightarrow [0, +\infty)$  by:

$$(*) \quad \rho'(v) = \sum_{g \in F} \sum_{w \cap gv \neq \emptyset} \rho(w).$$

Let  $\gamma \in \mathcal{F}_{\delta,r}(\partial P)$  and  $\theta \subset \bigcup_{g \in F} g\gamma$  such that  $\theta \in \mathcal{U}_\epsilon(\eta)$ . Then

$$L_{\rho'}(\gamma) = \sum_{g \in F} \sum_{v \cap \gamma \neq \emptyset} \sum_{w \cap gv \neq \emptyset} \rho(w) \geq \sum_{g \in F} \sum_{w \cap g\gamma \neq \emptyset} \rho(w).$$

However

$$L_\rho(\theta) \leq \sum_{g \in F} L_\rho(g\gamma) = \sum_{g \in F} \sum_{w \cap g\gamma \neq \emptyset} \rho(w).$$

Thus  $L_{\rho'}(\gamma) \geq L_\rho(\theta)$  and  $\rho'$  is  $\mathcal{F}_{\delta,r}(\partial P)$ -admissible.

Then the number of terms in the right-hand side of the definition (\*) is bounded by a constant  $N$  depending on  $\#F$ , the bi-Lipschitz constants of the elements of  $F$ , and the doubling constant of  $\partial\Gamma$ . Therefore by convexity

$$M_p(\rho') = \sum_{v \in G_k} \left( \sum_{g \in F} \sum_{w \cap gv \neq \emptyset} \rho(w) \right)^p \leq N^{p-1} \cdot \sum_{v \in G_k} \sum_{g \in F} \sum_{w \cap gv \neq \emptyset} \rho(w)^p \leq N^p \cdot \#F \cdot \sum_{w \in G_k} \rho(w)^p.$$

Which proves the inequality. The multiplicative constant induced by this part depends on  $p$ ,  $\eta$  and  $r$  as the set  $F$  depends on  $\eta$  and  $r$ . However it does not depend on  $k$ .

v) Here we write  $\delta(r, \partial P)$  for the constant  $\delta$  chosen in the previous part such that the curves of  $\mathcal{F}_{\delta,r}(\partial P)$  are  $(L, I)$ -curves for a certain  $L > 0$ .

By Proposition 7.20 there exist only a finite number of parabolic limit set of diameter larger than  $d_0$ . Moreover we recall that according to our notation, if  $\delta \geq \delta'$  then  $\mathcal{F}_{\delta',r}(\partial P) \subset \mathcal{F}_{\delta,r}(\partial P)$ . As a consequence, the function defined by

$$\delta_0(r) = \min \{ \delta(r, \partial P) : \text{diam } \partial P \geq d_0 \},$$

satisfies the desired property.

vi) As we see at the end of part iv),  $C = \lambda \cdot N^p$  with  $\lambda$  and  $N$  independent of  $p$ . Hence if  $p$  belongs to a compact subset  $K \subset [1, +\infty)$ , the constant  $C$  may be chosen independent of  $p$  by taking  $C = \lambda \cdot N^{\max K}$ .

□

As an immediate application, we notice that under the assumptions of the theorem, the behavior of  $\text{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k)$  as  $k$  goes to infinity does not depend, on the choice of  $\eta$  and



$\epsilon$ . Indeed, for  $p \geq 1$ ,  $r > 0$  and  $\delta < \delta_0(r)$  fixed, if  $\eta, \eta' \subset \partial P$  and  $\epsilon, \epsilon' > 0$  are such that the hypothesis of the theorem are satisfied. Then there exist  $C = C(\eta, \epsilon)$  and  $C' = C'(\eta', \epsilon')$  such that

$$C^{-1} \cdot \text{Mod}_p(\mathcal{U}_{\epsilon'}(\eta'), G_k) \leq \text{Mod}_p(\mathcal{U}_{\epsilon}(\eta), G_k) \leq C' \cdot \text{Mod}_p(\mathcal{U}_{\epsilon'}(\eta'), G_k).$$

Of course, if  $\eta = \eta'$  and  $\epsilon' < \epsilon$  we can choose  $C = 1$ .

### 6.3 Application to Fuchsian buildings

Another consequence of the theorem is that if the boundary of a graph product does not contain connected parabolic limit sets, then it satisfies the CLP.

**Theorem 6.13.** *Let  $\Gamma$  be a thick hyperbolic graph product such that  $\partial\Gamma$  is connected and any proper parabolic limit set is disconnected. Then  $\partial\Gamma$  equipped with a visual metric satisfies the CLP.*

*Proof.* We check the hypothesis of Proposition 3.12. To prove that  $\text{Mod}_1(\mathcal{F}_0, G_k)$  is unbounded, it is enough to check that for every  $N \in \mathbb{N}$  there exist  $N$  disjoint curves of diameter larger than  $d_0$  in  $\partial\Gamma$ . Indeed, this implies that for  $k \geq 0$  large enough  $\text{Mod}_1(\mathcal{F}_0, G_k) > N$ .

To prove this we use the assumption on the thickness which implies that for every  $N \in \mathbb{N}$  there exist  $N$  apartments with disjoint boundaries that intersects in a compact domain inside the building. To observe such apartments we use the following notation.

- $W_{i-1} = \{w \in \Gamma : w = s_1^i \dots s_k^i \text{ with } s_j \in S \text{ and } d_c(x_0, wx_0) = k\}$  for  $i = 1, 2$ .
- $\text{CB}_n = \text{Conv}(B_c(x_0, n)) \subset \text{Ch}(\Sigma)$  the convex hull for the metric over the chambers  $d_c(\cdot, \cdot)$  of  $B_c(x_0, n) \subset \text{Ch}(\Sigma)$  the ball of center  $x_0$  and of radius  $n$  for  $d_c(\cdot, \cdot)$  for  $n \geq 0$ .
- $\text{FCB}_n$  is the frontier of  $\text{CB}_n$ . By this we mean the set of all the chambers  $x \in \text{CB}_n$  such that there exists  $y \notin \text{CB}_n$  with  $x \sim y$ .

Then the following sets of chambers define  $N$  apartments all containing the base chamber  $x_0$ , and intersecting only inside  $\text{CB}_N$

- $A_1 = \{wx_0 : w \in W_0\}$ ,
- $A_2 = \{wx_0 : w \in W_1\}$ ,
- $A_3 = A_2 \cap \text{CB}_1 \cup \{gwx_0 : w \in W_0, gx_0 \in A_2 \cap \text{FCB}_1\}$ ,
- $A_n = A_{n-1} \cap \text{CB}_{n-2} \cup \{gwx_0 : w \in W_{n+1 \bmod(2)} \text{ and } gx_0 \in A_{n-1} \cap \text{FCB}_{n-2}\}$  for  $n = 2, \dots, N$ .

Now let  $\eta$  be a non-constant curve in  $\partial\Gamma$ . Up to a change of scale, by Proposition 3.11, we can assume  $\eta \in \mathcal{F}_0$ . Then as  $\partial\Gamma$  is the only parabolic limit set containing  $\eta$ , it is enough to apply Theorem 6.12 to satisfy the second hypothesis of Proposition 3.12.  $\square$

In particular, we can apply this result to the case of right-angled Fuchsian buildings. In the following, we call *right-angled Fuchsian building* a building associated with a graph product  $(C_n, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$  where  $C_n$  is the cyclic graph with  $n \geq 5$  vertices and  $q_1, \dots, q_n$  is a family of integers larger than or equal to 3.

**Corollary 6.14.** *For  $n \geq 5$ , let  $C_n$  be the cyclic graph with  $n$  vertices and let  $q_1, \dots, q_n$  be a family of integers larger than or equal to 3. Let  $\Gamma$  be the graph product given by the pair  $(C_n, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$ . Then  $\partial\Gamma$  equipped with a visual metric satisfies the CLP.*

This result was known since boundaries of right-angled Fuchsian buildings are Loewner spaces (see [BP00, Proposition 2.3.4.]). However, here we give a direct proof of this result.

Furthermore, we can prove that these thick graph products are the only ones that satisfy the hypothesis of Theorem 6.13. To verify this we need to introduce the following simplicial complex.

**Definition 6.15.** *Let  $\Gamma = (\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$  be a graph product, the nerve of  $\Gamma$  is the simplicial complex  $L = L(\mathcal{G})$  such that:*

- *the 1-skeleton of  $L$  is  $\mathcal{G}$ ,*
- *$k$  vertices of  $\mathcal{G}$  span a  $(k-1)$ -simplex in  $L$  if and only if the corresponding parabolic subgroup in  $\Gamma$  is finite.*

For a simplex  $\sigma \subset L$  spanned by the vertices  $v_1, \dots, v_k \in \mathcal{G}^{(0)}$ , we denote by  $\mathcal{G}_\sigma$  the full sub-graph of  $\mathcal{G}$  spanned by the vertices  $\mathcal{G}^{(0)} \setminus \{v_1, \dots, v_k\}$ . Now we write  $L \setminus \sigma = L(\mathcal{G}_\sigma)$ . Note that  $L \setminus \sigma$  can be seen as a subcomplex of  $L$ .

The following theorem is a special case of [DM02, Corollary 5.14.].

**Theorem 6.16.** *The boundary of  $\Gamma$  is connected if and only if the subcomplex  $L \setminus \sigma$  is connected for any simplex  $\sigma \subset L$ .*

Now we can prove the following.

**Proposition 6.17.** *Let  $\Gamma = (\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$  be a hyperbolic graph product. Assume that  $\partial\Gamma$  is connected and that any proper parabolic limit set  $\partial P$  is disconnected, then the building associated with  $\Gamma$  is a right-angled Fuchsian building.*

*Proof.* We only need to prove that  $\mathcal{G}$  contains a circuit of length  $n \geq 5$ . According to Corollary 6.7, if any proper parabolic limit set in  $\partial\Gamma$  is disconnected then any proper parabolic limit set in  $\partial\Gamma$  is discrete. Moreover,  $\partial\Gamma$  contains at least one proper parabolic

limit set of the form  $\partial\Gamma_I$  with  $\#I = n - 1$  otherwise  $\partial\Gamma = \emptyset$ . The subgroup  $\Gamma_I$  is a graph product associated with the graph  $\mathcal{G}_I$ . This graph is obtained from  $\mathcal{G}$  to which we remove a vertex  $p$  and all the edges adjacent to  $p$ . Then if  $L_I$  is the nerve associated with  $\Gamma_I$ , we get  $L_I$  from  $L$  to which we remove the interior of any simplex containing  $p$ .

Now, thanks to Theorem 6.16, we know that there exists a simplex  $\sigma \subset L_I$  such that  $L_I \setminus \sigma$  is disconnected. Let  $C_1$  and  $C_2$  be two connected components of  $L_I \setminus \sigma$ . Up to a subsimplex, we can assume that any vertex of  $\sigma$  is connected to  $C_1$  or to  $C_2$  by an edge. However, if we consider the simplex  $\sigma$  in  $L$ , we see that  $L \setminus \sigma$  is connected because  $\partial\Gamma$  is connected. Therefore there exist at least one edge attaching  $p$  to  $C_1$  and at least one edge attaching  $p$  to  $C_2$ .

We set  $V = \{v_1, \dots, v_k\}$  the vertices of  $\sigma$  that are not connected to  $p$  by an edge and  $V' = \{v'_1, \dots, v'_{k'}\}$  the rest of the vertices of  $\sigma$ . At this point, we assume by contradiction, that  $\mathcal{G}$  contains no circuit of length  $n \geq 4$ . We can check that under this assumption the following situations does not occur

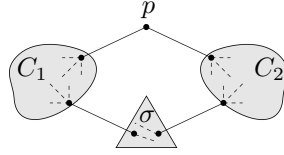


Figure 3: Forbidden situation *i)*

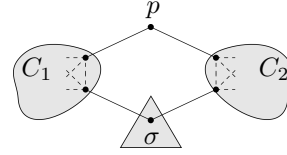


Figure 4: Forbidden situation *ii)*

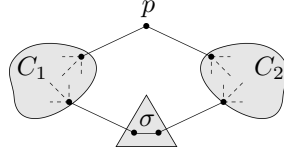


Figure 5: Forbidden situation *iii)*

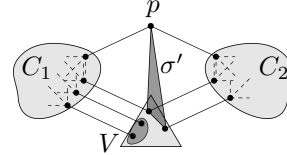


Figure 6: Resulting situation

- i)  $V'$  is empty,
- ii) there exists  $v \in V$  such that  $v$  is adjacent to both  $C_1$  and  $C_2$ ,
- iii) there exist  $v, w \in V$  such that  $v$  is adjacent to  $C_1$  and  $w$  is adjacent to  $C_2$ .

Hence, the vertices in  $V$  are either all adjacent to  $C_1$  or all adjacent to  $C_2$ . Assume that the vertices in  $V$  are all adjacent to  $C_1$ . As a consequence, if  $\sigma'$  designates the simplex in  $L$  spanned by  $V' \cup \{p\}$  then  $L \setminus \sigma'$  is not connected. This is not possible because  $\partial\Gamma$  is connected.

Therefore  $\mathcal{G}$  contains a circuit of length  $n \geq 4$ , but as  $\Gamma$  is hyperbolic it contains no circuit of length 4. This concludes the proof.  $\square$

## 7 Combinatorial metric on boundaries of right-angled hyperbolic buildings

In this section we explain how the geometry of the boundary is determined by boundaries of building-walls. We start by discussing the geometry of intersections of dials of building and the boundaries of such intersections. Then, we describe a combinatorial and self-similar metric on  $\partial\Gamma$  in terms of dials of building. Finally, we construct an approximation of  $\partial\Gamma$  that will be convenient to use in Section 8.

Here we use the notation and assumptions of Section 5 and 6. In particular,  $\Gamma$  is a fixed graph product given by the pair  $(\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$  and acting on the building  $\Sigma$ . The base chamber is  $x_0$ , and  $W$  is the right-angled Coxeter group associated with  $\Gamma$ . We assume that  $\Gamma$  is hyperbolic and  $\partial\Gamma$  is connected.

### 7.1 Projections of chambers in $\Sigma$

In right-angled buildings, we can project chambers on residues and on dials of building. This will be useful in the rest of this section to understand the metric on the boundary in the hyperbolic case.

**Proposition 7.1.** *Let  $D$  be a residue or a dial of building and  $C = \text{Ch}(D)$ . Then for any  $x \in \text{Ch}(\Sigma)$  there exists a unique chamber  $\text{proj}_C(x) \in C$  such that*

$$d_c(x, \text{proj}_C(x)) = \text{dist}(x, C).$$

*Moreover, for any chamber  $y \in C$  there exists a minimal gallery from  $x$  to  $y$  passing through  $\text{proj}_C(x)$ .*

For simplicity, in the following we sometime use the notation  $\text{proj}_D(\cdot)$  to designate  $\text{proj}_{\text{Ch}(D)}(\cdot)$ . Before the proof of the proposition we need to establish the following lemma.

**Lemma 7.2.** *Let  $M$  and  $M'$  be two building-walls such that  $M \perp M'$ . Let  $r \in \Gamma$  be a rotation around  $M$  and  $D'$  be a dial of building bounded by  $M'$  then  $r(D') = D'$ .*

*Proof.* Up to a translation on the dials and a conjugation on the rotations we can assume that  $M$  and  $M'$  are along  $x_0$  and  $x_0 \subset D'$ . If  $s$  is a rotation around  $M'$ , we can write

$$\text{Ch}(D') = \{x \in \text{Ch}(\Sigma) : d_c(x_0, x) < d_c(x_0, sx)\}.$$

Hence

$$\begin{aligned} r(\text{Ch}(D')) &= \{rx \in \text{Ch}(\Sigma) : d_c(x_0, x) < d_c(x_0, sx)\} \\ &= \{x \in \text{Ch}(\Sigma) : d_c(x_0, r^{-1}x) < d_c(x_0, sr^{-1}x)\}. \end{aligned}$$

By assumption  $rs = sr$ , thus  $r(\text{Ch}(D')) = \{x \in \text{Ch}(\Sigma) : d_c(rx_0, x) < d_c(rx_0, sx)\}$ . Moreover,  $d_c(x_0, rx_0) = 1$  and  $d_c(x_0, srx_0) = 2$  so  $rx_0 \in \text{Ch}(D')$ . With Fact 5.20 we obtain  $\text{Ch}(D') = r(\text{Ch}(D'))$ .  $\square$

*Proof of Proposition 7.1.* If  $D$  is a residue, then we refer to [Tit74, Proposition 3.19.3]. If  $D$  is a dial of building, let  $y \in C$  be such that  $d_c(x, y) = \text{dist}(x, C)$ . Then for  $z \in C$  we set  $x = x_1 \sim x_2 \sim \dots \sim y$  and  $y = x_\ell \sim \dots \sim z = x_k$  two minimal galleries. Assume that the gallery

$$x = x_1 \sim x_2 \sim \dots \sim y = x_\ell \sim \dots \sim z = x_k$$

is not minimal. Then there exists a building-wall  $M$  and two indices  $i, j$  with  $1 \leq i < \ell$  and  $\ell \leq j < k$  such that

- $M$  separates  $x_i$  and  $x_{i+1}$ ,
- $M$  separates  $x_j$  and  $x_{j+1}$ .

Now consider  $r \in \Gamma$  the rotation around  $M$  such that  $rx_{i+1} = x_i$ . By Proposition 7.2,  $r(D) = D$  hence, the gallery

$$x \sim \dots \sim x_i \sim rx_{i+2} \dots \sim rx_\ell = ry$$

connects  $x$  to  $C$  and is of length  $\text{dist}(x, C) - 1$ , which is a contradiction.

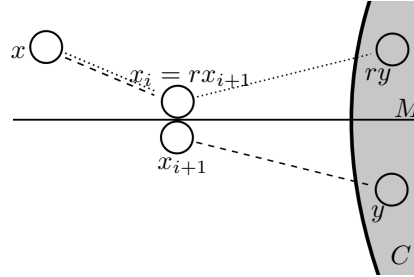


Figure 7

We proved that for any  $z \in C$ , there exists a minimal gallery from  $x$  to  $z$  passing through  $y$ . This proves in particular that  $y$  is unique and the proof is achieved.  $\square$

The following lemma says that the projections on the dials of building are orthogonal relatively to the building-wall structure.

**Lemma 7.3.** *Let  $D, D' \in \mathcal{D}(\Sigma)$  such that  $\text{Ch}(D \cap D') \neq \emptyset$ . If  $x \in \text{Ch}(D)$  then  $\text{proj}_{D'}(x) \in \text{Ch}(D \cap D')$ .*

*Proof.* Clearly  $\text{proj}_{D'}(x) \subset D'$  so we check that  $\text{proj}_{D'}(x) \subset D$ . Under the assumption  $\text{Ch}(D \cap D') \neq \emptyset$  three cases are possible. First, if  $D' \subset D$  then  $\text{proj}_{D'}(x) \subset D' \subset D$ . Then if  $D \subset D'$ , then  $\text{proj}_{D'}(x) = x \subset D$  for any  $x \in \text{Ch}(D)$ .

Now let  $M$  and  $M'$  be the building-walls that bound  $D$  and  $D'$ . The last case is realized when  $M \perp M'$ . In this case consider a minimal gallery  $x \sim \cdots \sim \text{proj}_{D'}(x)$ .

If  $\text{proj}_{D'}(x) \not\subset D$ , then the preceding gallery crosses  $M$ . As a consequence, we can write that there exists a minimal gallery of the form

$$x \sim \cdots \sim x_i \sim x_{i+1} = rx_i \sim x_{i+2} \sim \cdots \sim \text{proj}_{D'}(x)$$

where  $r \in \Gamma$  is rotation around  $M$ . Then with Lemma 7.2 we obtain that

$$x \sim \cdots \sim x_i \sim r^{-1}x_{i+2} \sim \cdots \sim r^{-1}\text{proj}_{D'}(x)$$

is a gallery between  $x$  and  $D'$  of length  $d_c(x, \text{proj}_{D'}(x)) - 1$ . Which is a contradiction.  $\square$

Applying several times the projection maps on dials of building, we define projection maps on finite intersections of dials of building.

**Proposition 7.4.** *Let  $D_1, \dots, D_k \in \mathcal{D}(\Sigma)$  and  $C = \text{Ch}(D_1 \cap \cdots \cap D_k)$ . Assume that  $C \neq \emptyset$ . Then for any  $x \in \text{Ch}(\Sigma)$  there exists a unique chamber  $\text{proj}_C(x) \in C$  such that*

$$d_c(x, \text{proj}_C(x)) = \text{dist}(x, C).$$

*Moreover, for any chamber  $y \in C$  there exists a minimal gallery from  $x$  to  $y$  passing through  $\text{proj}_C(x)$ . Finally  $\text{proj}_C(x) = \text{proj}_{D_k} \circ \cdots \circ \text{proj}_{D_1}(x)$ .*

For simplicity, in the following we will use the notation  $\text{proj}_D(\cdot)$  instead of  $\text{proj}_C(\cdot)$ . Notice that it is not always possible to define a projection on a convex set of chambers. For instance, if  $\Sigma$  is a thick building there exist pairs of adjacent chambers  $x$  and  $y$  with  $d_c(x_0, x) = d_c(x_0, y)$ .

*Proof.* First, according to Lemma 7.3, we can assume, up to a subfamily, that  $x \not\subset D_i$  for each  $i = 1, \dots, k$ . Now we set

- $C_1 = \text{Ch}(D_1)$  and  $C_i = C_{i-1} \cap \text{Ch}(D_i)$  for any  $i = 2, \dots, k$ ,
- $x_1 = \text{proj}_{D_1}(x)$  and  $x_i = \text{proj}_{D_i}(x_{i-1})$  for any  $i = 2, \dots, k$ .

By induction on  $i$  we prove the following property:

$x_i \in C_i$  and is the unique chamber of  $C_i$  such that  $d_c(x, x_i) = \text{dist}(x, C_i)$ .  
Moreover, for any chamber  $y \in C_i$  there exists a minimal gallery from  $x$  to  $y$  passing through  $x_i$ .

If  $i = 1$  the property holds by Proposition 7.1. Let  $i > 1$  and assume that the property holds at rank  $i$ . In particular, for  $j = 1, \dots, i$  one has  $x_i \in \text{Ch}(D_j)$  and  $\text{Ch}(D_j) \cap \text{Ch}(D_{i+1}) \neq \emptyset$ . Therefore, with Lemma 7.3,  $x_{i+1} \in \text{Ch}(D_1) \cap \dots \cap \text{Ch}(D_i) \cap \text{Ch}(D_{i+1}) = C_{i+1}$ .

By Proposition 7.1,  $x_{i+1}$  is the unique chamber in  $C_{i+1}$  such that

$$d_c(x_i, x_{i+1}) = \text{dist}(x_i, C_{i+1}).$$

Moreover, by the same proposition, for  $y \in C_i$  if the galleries  $x \sim \dots \sim x_i$  and  $x_i \sim \dots \sim y$  are minimal, the gallery

$$x \sim \dots \sim x_i \sim \dots \sim y$$

is minimal. As a consequence,  $x_{i+1}$  is the unique chamber in  $C_{i+1}$  such that  $d_c(x, x_{i+1}) = \text{dist}(x, C_{i+1})$ .

On the other hand, by Proposition 7.1, for any  $y \in C_{i+1}$  there exists a minimal gallery

$$x_i \sim \dots \sim x_{i+1} \sim \dots \sim y.$$

Hence if the gallery  $x \sim \dots \sim x_i$  is minimal, the gallery

$$x \sim \dots \sim x_i \sim \dots \sim x_{i+1} \sim \dots \sim y$$

is also minimal. □

## 7.2 Shadows on $\partial\Gamma$

The following notions are used in the rest of this article to describe the topology and the metric on  $\partial\Gamma$ . We recall that the boundary of  $\Gamma$  is canonically identified with the boundary of  $\Sigma$ .

**Definition 7.5.** *Let  $x \in \text{Ch}(\Sigma)$ . We call cone of chambers of base  $x$  and we denote  $C_x \subset \Sigma$ , the union of the set of chambers  $y \in \text{Ch}(\Sigma)$  such that there exists a minimal gallery from  $x_0$  to  $y$  passing through  $x$ .*

Cones of chambers are characterized by projection maps and dials of building.

**Proposition 7.6.** *Let  $D_1, \dots, D_k \in \mathcal{D}(\Sigma)$  and  $C = D_1 \cap \dots \cap D_k$ . Assume that  $C$  contains a chamber and that  $x_0 \notin D_i$  for  $i = 1, \dots, k$ . If we set  $x = \text{proj}_C(x_0)$  then  $C_x = C$ .*

*Proof.* According to Definition 7.5 and to Proposition 7.4,  $C \subset C_x$ . Now let  $y \in \text{Ch}(C_x)$  and let  $M_i$  be the building-wall that bounds  $D_i$  for each  $i = 1, \dots, k$ . If  $x_0 \sim \dots \sim x \sim \dots \sim y$  is a minimal gallery, then the subgallery  $x \sim \dots \sim y$  does not cross the building-wall  $M_i$  for any  $i = 1, \dots, k$  and  $y \subset C$ . □

Reciprocally cones are intersections of dials of building.

**Proposition 7.7.** *Let  $x \in \text{Ch}(\Sigma)$  and let  $D_1, \dots, D_k \in \mathcal{D}(\Sigma)$  be the family of dials of building such that for any  $i = 1, \dots, k$*

$$x_0 \notin D_i \text{ and } x \subset D_i.$$

*Then  $C_x = D_1 \cap \dots \cap D_k$ .*

*Proof.* Let  $C = D_1 \cap \dots \cap D_k$ . According to Proposition 7.6, it is enough to prove that  $\text{proj}_C(x_0) = x$ . If we write  $x' = \text{proj}_C(x_0)$ , with Proposition 7.6,  $C = C_{x'}$ . Hence there exists a minimal gallery

$$x_0 \sim \dots \sim x' \sim \dots \sim x.$$

Now assume that  $x' \neq x$ , this means that there exists a building-wall  $M$  that separates  $x$  and  $x'$ . Since the preceding gallery is minimal, the dial of building  $D$  bounded by  $M$  that contains  $x$  does not contain  $x'$  and  $x_0$ . Thus  $D \in \{D_1, \dots, D_k\}$  and  $x' \not\subset C$  which is a contradiction.  $\square$

In particular, as cones of chambers are intersections of dials of building, it makes sense to consider projection maps on cones and  $\text{proj}_{C_x}(x_0) = x$ . In the following fact we summarize what we can say about intersections of dials of building.

**Fact 7.8.** *Let  $D_1, \dots, D_k \in \mathcal{D}(\Sigma)$  be such that  $D_i \not\subset D_j$  for any  $i \neq j$ . Assume that  $x_0 \notin D_i$  for any  $i = 1, \dots, k$ . Let  $M_i$  be the building-wall that bounds  $D_i$  for any  $i = 1, \dots, k$ , and let  $C = D_1 \cap \dots \cap D_k$ . Then exactly one of these assertions holds.*

- *There exists  $i, j$  such that  $M_i \parallel M_j$  and  $C = \emptyset$ .*
- *$M_i \cap M_j \neq \emptyset$  for any  $i, j = 1, \dots, k$ , and there exists  $i \neq j$  such that  $M_i = M_j$ . In this case  $C$  is contained in  $M_i$ .*
- *$M_i \perp M_j$  for any  $i \neq j$ . In this case  $C$  is a cone.*

This fact, up to a translation and up to a subfamily, describes how a finite family of dials intersects. The following lemma specifies the case when the intersection is a cone.

**Lemma 7.9.** *Let  $D_1, \dots, D_k$  be a family of distinct dials of building bounded by the building-walls  $M_1, \dots, M_k$ . Let  $C = D_1 \cap \dots \cap D_k$ . Assume that  $x_0 \notin D_i$  and that  $M_i \perp M_j$  for any  $i, j = 1, \dots, k$  with  $i \neq j$ . Then any  $M_i$  is along  $\text{proj}_C(x_0)$ .*

*Proof.* First if  $k = 1$  the property is clearly satisfied. According to Lemma 7.3,  $\text{proj}_{D_1}(x_0) \notin \text{Ch}(D_2) \cup \dots \cup \text{Ch}(D_k)$ . Applying this lemma  $k - 2$  times we obtain that

$$\text{proj}_{D_{k-1}} \circ \dots \circ \text{proj}_{D_1}(x_0) \notin \text{Ch}(D_k).$$

Hence  $\text{proj}_C(x_0) = \text{proj}_{D_k}(\text{proj}_{D_{k-1}}(\dots \circ \text{proj}_{D_1}(x_0)))$  is along  $M_k$ . Finally, we apply the same argument to  $M_1, \dots, M_{k-1}$  and the proof is finished.  $\square$



Finally we obtain the following characterization of cones.

**Proposition 7.10.** *Let  $x \in \text{Ch}(\Sigma)$  and let  $D_1, \dots, D_k$  be the family of dials of building bounded by  $M_1, \dots, M_k$  such that for any  $i = 1, \dots, k$*

$$x_0 \notin D_i, \ x \subset D_i, \text{ and } M_i \text{ is along } x.$$

*Then*

$$i) \ C_x = D_1 \cap \dots \cap D_k,$$

ii) *there exist  $g \in \Gamma$  and  $M_1^0, \dots, M_k^0 \in \mathcal{M}(\Sigma)$  with*

- *for any  $i \neq j$ :  $M_i^0 \perp M_j^0$ ,*
- *for any  $i$ :  $M_i^0$  is along  $x_0$ ,*

*such that  $C_x = g(C)$  where  $C = D_0(M_1') \cap \dots \cap D_0(M_k')$ .*

*Proof.* Let  $D'_1, \dots, D'_\ell$  be the family of dials of building such that  $x_0 \notin D_i$  and  $x \subset D_i$  for any  $i = 1, \dots, \ell$ . Then

$$\{D_1, \dots, D_k\} \subset \{D'_1, \dots, D'_\ell\}.$$

According to Proposition 7.7,  $C_x = D'_1 \cap \dots \cap D'_\ell$ , thus  $C_x \subset D_1 \cap \dots \cap D_k$ . Let  $M'_i$  be the wall that bounds  $D'_i$  for any  $i = 1, \dots, \ell$ . Up to a subfamily, we can assume that  $C_x = D'_1 \cap \dots \cap D'_\ell$  with  $M'_i \perp M'_j$  for any  $i \neq j$ . Then, as  $\text{proj}_{C_x}(x_0) = x$ , by Lemma 7.9, any building-wall  $M'_i$  is along  $x$ . Finally, we get

$$\{D'_1, \dots, D'_\ell\} \subset \{D_1, \dots, D_k\}$$

and  $D_1 \cap \dots \cap D_k \subset C_x$ .

The second part is immediate with  $g \in \Gamma$  such that  $gx_0 = x$ .

□

The following lemma is used to prove that boundaries of cones are of non-empty interior.

**Lemma 7.11.** *Let  $M_1, \dots, M_k$  be a collection of building-walls such that any  $M_i$  admits a parallel building-wall. Assume that  $M_i \perp M_j$  for any  $i \neq j$ . Then there exists  $M \in \mathcal{M}(\Sigma)$  such that  $M \parallel M_i$  for any  $i$ .*

*Proof.* We prove the proposition by induction on  $k$ . For  $k = 1$  there is nothing to prove. For  $k \geq 2$ , pick a collection of building-walls  $M_1, \dots, M_k$  satisfying the hypothesis of the lemma. Assume that there exists  $M$  a building-wall such that  $M \parallel M_1, \dots, M \parallel M_k$ . Let  $M_{k+1} \in \mathcal{M}(\Sigma)$  be such that  $M_{k+1} \perp M_1, \dots, M_{k+1} \perp M_k$ .

If  $M$  is parallel to  $M_{k+1}$  there is nothing more to say. Now we assume  $M \perp M_{k+1}$  and we pick a wall  $M'$  parallel to  $M_{k+1}$ . If  $M'$  is parallel to  $M_1, \dots, M_k$  there is nothing more to say. Now we assume that there exists  $1 \leq i \leq k$  such that, up to a reordering

$$M' \perp M_1, \dots, M' \perp M_i, M' \parallel M_{i+1}, \dots, M' \parallel M_k, M' \parallel M_{k+1}.$$

First we consider the case  $M' \perp M$ . With

- $M \perp M_{k+1}, M_{k+1} \perp M_1, M_1 \perp M'$ ,
- and  $M' \parallel M_{k+1}, M \parallel M_1$ ,

we obtain that the building-walls  $M', M, M_{k+1}$ , and  $M_1$  form a right-angled rectangle. Which is a contradiction with the hyperbolicity of  $\Sigma$ .

Secondly we consider the case  $M' \parallel M$ . Let  $r' \in \Gamma$  be a rotation around  $M'$ . Then  $r'(M)$  is such that

$$r'(M) \parallel M_1, \dots, r'(M) \parallel M_i.$$

Indeed, as  $M \parallel M_j$ , it comes that  $r'(M) \parallel r'(M_j)$  and, as  $M' \perp M_j$ , by Lemma 7.2,  $r'(M_j) = M_j$  for  $1 \leq j \leq i$ . Thus  $r'(M) \parallel M_j$ .

Moreover

$$r'(M) \parallel M_{i+1}, \dots, r'(M) \parallel M_{k+1}.$$

Indeed, as  $M_{i+1} \cap \dots \cap M_{k+1} \neq \emptyset$  and  $M' \parallel M_j$  for  $i+1 \leq j \leq k+1$ , the building-walls  $M_{i+1}, \dots, M_{k+1}$  are entirely contained in the same connected component of  $\Sigma \setminus M'$ . Let  $C$  be this connected component. Since  $M \parallel M'$  and  $M \cap M_{k+1} \neq \emptyset$  it comes that  $M$  is also contained in  $C$ . Thus  $r'(M)$  is not contained in  $C$  and  $r'(M) \parallel M_j$  for  $i+1 \leq j \leq k+1$ .  $\square$

**Proposition 7.12.** *Let  $x \in \text{Ch}(\Sigma)$  and  $C_x$  be the cone based at  $x$ . Then  $\partial C_x$  is of non-empty interior in  $\partial \Gamma$ .*

*Proof.* By Fact 7.8 and Proposition 7.10 we can write

$$\partial C_x = \partial D_1 \cap \dots \cap \partial D_k,$$

where  $D_1, \dots, D_k$  is a collection of dials of building bounded by the building-walls  $M_1, \dots, M_k$  with  $M_i \perp M_j$  for any  $i \neq j$ .

Up to a subfamily, we can assume that for every  $i = 1, \dots, k$  there exists  $M'_i \in \mathcal{M}(\Sigma)$  such that  $M_i \parallel M'_i$ . Indeed if  $M$  admits no parallel building wall then  $\partial M = \partial \Gamma$  (see beginning of Section 5.10).

By rotations around  $M_1, \dots, M_k$ , all the connected components of  $\Sigma \setminus (M_1 \cup \dots \cup M_k)$  are isomorphic. Hence, thanks to Lemma 7.11, there exists  $M \in \mathcal{M}(\Sigma)$  such that

$$M \parallel M_i \text{ for any } i = 1, \dots, k \text{ and } M \subset D_1 \cap \dots \cap D_k.$$

In particular, there exists  $D \in \mathcal{D}(\Sigma)$  bounded by  $M$  such that  $D \subset D_1 \cap \dots \cap D_k$ . As  $\partial D$  is of non-empty interior, we obtain that  $\partial C_x$  is of non-empty interior.  $\square$

In the rest of this article, we use boundaries of cones as a base for the topology of  $\partial\Gamma$  and to construct approximations.

**Definition/Notation 7.13.** *Let  $x \in \text{Ch}(\Sigma)$  and  $C_x$  be the corresponding cone of chambers. We call shadow of  $x$  the boundary of  $C_x$  in  $\partial\Gamma$  and we write  $v_x = \partial C_x$ .*

### 7.3 Combinatorial metric on $\partial\Gamma$

Until now we have been considering on  $\partial\Gamma$  the visual metric coming from the geometric action of  $\Gamma$  on  $\Sigma$ . Now we use minimal galleries to describe a combinatorial metric on  $\partial\Gamma$  that will be more convenient to use in the sequel. We extend the notion of minimal gallery to infinite galleries.

**Definition 7.14.** *An infinite gallery  $x_0 \sim x_1 \sim \dots$  (resp. a bi-infinite gallery  $\dots \sim x_{-1} \sim x_0 \sim x_1 \sim \dots$ ) is minimal if for any  $k \in \mathbb{N}$  (resp.  $k \in \mathbb{Z}$ ) and  $\ell \in \mathbb{N}$  the gallery  $x_k \sim \dots \sim x_{k+\ell}$  is minimal.*

Let  $\mathcal{DG}(\Sigma)$  denote the *dual graph* of  $\Sigma$ . This graph is defined by:

- The set of vertices  $\mathcal{DG}(\Sigma)^{(0)}$  is given by  $\text{Ch}(\Sigma)$  the set of chambers in  $\Sigma$ . If  $v \in \mathcal{DG}(\Sigma)^{(0)}$  then  $c_v$  denotes the corresponding chamber in  $\text{Ch}(\Sigma)$ .
- There exists an edge between two vertices  $v_1$  and  $v_2$  if and only if  $c_{v_1}$  is adjacent to  $c_{v_2}$  in  $\Sigma$ .
- Each edge is isometric to the segment  $[0, 1]$ .

Naturally,  $\mathcal{DG}(\Sigma)$  is a proper geodesic and hyperbolic space. It is quasi-isometric to  $\Sigma$  and the action of  $\Gamma$  on  $\mathcal{DG}(\Sigma)$  is geometric. Therefore we identify

$$\partial\mathcal{DG}(\Sigma) \simeq \partial\Gamma.$$

**Example 7.15.** *We recall that the group  $\Gamma$  is given by the following presentation*

$$\Gamma = \langle s_i \in S \mid s_i^{q_i} = 1, s_i s_j = s_j s_i \text{ if } v_i \sim v_j \rangle.$$

*If  $q_i = 2$  or  $3$  for any  $i = 1, \dots, n$  then  $\mathcal{DG}(\Sigma)$  is identified with  $\text{Cay}(\Gamma)$  the Cayley graph of  $\Gamma$  with respect to the recalled generating set. Otherwise, if we consider a generator  $s \in S$  of  $\Gamma$  of order  $q \geq 4$  then in  $\mathcal{DG}(\Sigma)$  the full sub-graph generated by the vertices associated with  $e, s, \dots, s^{q-1}$  is a complete graph. In  $\text{Cay}(\Gamma)$  the full sub-graph generated by the vertices associated with  $e, s, \dots, s^{q-1}$  is a cyclic graph of length  $q$ . Nevertheless  $\mathcal{DG}(\Sigma)$  and  $\text{Cay}(\Gamma)$  are always quasi-isometric.*

With Definition 7.14, infinite minimal galleries are identified with geodesic rays in  $\mathcal{DG}(\Sigma)$  starting from a vertex. Therefore we can identify  $\partial\Gamma$  with the set of equivalence classes of infinite galleries starting at  $x_0$  where two such galleries  $x_0 \sim x_1 \sim \dots$  and  $y_0 = x_0 \sim y_1 \sim \dots$  are equivalent if and only if there exists  $K > 0$  such that  $d_c(x_i, y_i) < K$  for all  $i \in \mathbb{N}$ .

**Example 7.16.** Here we consider only minimal galleries. We write  $\mathcal{R}$  the equivalence relation on the infinite galleries starting at  $x_0$  defined above. Let  $x \in \text{Ch}(\Sigma)$  with  $d_c(x_0, x) = k \geq 1$ . Then we can describe the shadow  $v_x$  as follows

$$v_x \simeq \{x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots : x_i \in \text{Ch}(\Sigma) \text{ and } x_k = x\} / \mathcal{R}.$$

Likewise, if  $\partial P$  is a parabolic limit set associated with the residue  $g\Sigma_I$ . Let  $x := \text{proj}_{g\Sigma_I}(x_0)$  and assume that  $d_c(x_0, x) = k \geq 1$ . Then we can describe  $\partial P$  as follows

$$\partial P \simeq \{x_0 \sim x_1 \sim \cdots : x_k = x \text{ and } x_{k+i} \sim_{s_i} x_{k+i+1} \text{ with } s_i \in I \text{ for any } i \geq 0\} / \mathcal{R}.$$

Now we use the following notation.

**Notation.** If  $x_0 \sim x_1 \sim \cdots$  is a minimal infinite gallery that goes asymptotically to  $\xi \in \partial\Gamma$ , then we write

$$\xi = [x_0 \sim x_1 \sim \cdots].$$

**Definition 7.17.** Let  $\xi, \xi'$  be two distinct points in  $\partial\Gamma$ , let  $\{\xi|\xi'\}_{x_0}$  denote the largest integer  $\ell$  such that there exist two infinite minimal galleries representing  $\xi$  and  $\xi'$

$$\xi = [x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots] \text{ and } \xi' = [x_0 \sim x'_1 \sim \cdots \sim x'_i \sim \cdots]$$

with

$$x_i = x'_i \text{ for } i \leq \ell \text{ and } x_{\ell+1} \neq x'_{\ell+1}.$$

In terms of shadows,  $\{\xi|\xi'\}_{x_0}$  is the largest integer such that there exists a shadow  $v_x$ , with  $d_c(x_0, x) = \{\xi|\xi'\}_{x_0}$ , that contains both  $\xi$  and  $\xi'$ . The following proposition gives a characterization of this quantity in terms of building-walls. We recall that  $D_0(M)$  designates the dial of building bounded by  $M$  and containing  $x_0$ .

**Proposition 7.18.** Let  $\xi, \xi'$  be two distinct points in  $\partial\Gamma$ . Then

$$\{\xi|\xi'\}_{x_0} = \#\{M \in \mathcal{M}(\Sigma) : \text{there exists } \alpha \neq 0 \text{ s.t. } \{\xi, \xi'\} \subset \partial D_\alpha(M)\}.$$

*Proof.* Let  $M_1, \dots, M_k$  be the set of building-walls such that there exists  $\alpha \neq 0$  with  $\{\xi, \xi'\} \subset \partial D_\alpha(M)$ . Let  $\ell = \{\xi|\xi'\}_{x_0}$ . We prove that  $k = \ell$ .

For  $i = 1, \dots, k$ , let  $D_i$  be a dial of building bounding  $M_i$  such that  $\{\xi, \xi'\} \subset \partial D_i$  and  $x_0 \notin D_i$ . We set  $C = D_1 \cap \cdots \cap D_k$ . Since the building-walls are distinct and  $\partial C \neq \emptyset$  it follows from Fact 7.8 that  $C$  is a cone. Let  $x = \text{proj}_C(x_0)$ . As  $\{\xi, \xi'\} \subset \partial C$ , there exists an infinite minimal gallery starting from  $x_0$  going asymptotically to  $\xi$  (resp.  $\xi'$ ) passing through  $x$ . Finally, we obtain  $\ell \geq d_c(x_0, x) \geq k$ .

Now consider  $x_0 \sim x_1 \sim \cdots \sim x_i \sim \cdots$  (resp.  $x_0 \sim x'_1 \sim \cdots \sim x'_i \sim \cdots$ ) a minimal infinite gallery representing  $\xi$  (resp.  $\xi'$ ) in  $\partial\Gamma$ . Assume that

$$x_i = x'_i \text{ for } i \leq \ell \text{ and } x_{\ell+1} \neq x'_{\ell+1}.$$

For any  $i = 1, \dots, \ell$  let  $D'_i$  be the dial of building such that  $x_{i-1} \notin D'_i$  and  $x_i \in D'_i$ . By minimality of the galleries, we get that  $\{\xi, \xi'\} \subset \partial D'_i$  for any index  $i$ . Therefore  $\ell \leq k$  and the proof is finished.  $\square$

In the following, we prove that  $\{\cdot|\cdot\}_{x_0}$  coincides with a Gromov product in  $\partial\Gamma$  and thus controls a visual metric on  $\partial\Gamma$ .

**Proposition 7.19.** *Let  $\xi, \xi'$  be two distinct points in  $\partial\Gamma$ . Then there exists a bi-infinite minimal gallery between  $\xi$  and  $\xi'$  that lies at a distance smaller than  $\{\xi|\xi'\}_{x_0} + 1$  of  $x_0$ .*

*Proof.* Let  $\ell = \{\xi|\xi'\}_{x_0}$  and assume that  $\xi = [x_0 \sim x_1 \sim \dots \sim x_i \sim \dots]$  and  $\xi' = [x_0 \sim x'_1 \sim \dots \sim x'_i \sim \dots]$  with

$$x_i = x'_i \text{ for } i \leq \ell \text{ and } x_{\ell+1} \neq x'_{\ell+1}.$$

We consider two cases. Either  $x_{\ell+1}$  is adjacent to  $x'_{\ell+1}$ , or  $x_{\ell+1}$  is not adjacent to  $x'_{\ell+1}$ . In the first case, Proposition 5.23 implies that the bi-infinite gallery

$$\dots \sim x_{\ell+2} \sim x_{\ell+1} \sim x'_{\ell+1} \sim x'_{\ell+2} \sim \dots$$

only crosses once the building-walls that separate  $\xi$  and  $\xi'$ . Hence it is minimal.

In the second case, we apply the same reasoning to the bi-infinite gallery

$$\dots \sim x_{\ell+2} \sim x_{\ell+1} \sim x_\ell \sim x'_{\ell+1} \sim x'_{\ell+2} \sim \dots.$$

Finally  $\{\xi|\xi'\}_{x_0}$  or  $\{\xi|\xi'\}_{x_0} + 1$  is the distance between  $x_0$  and a bi-infinite minimal gallery between  $\xi$  and  $\xi'$ .  $\square$

**Notation.** Let  $d(\cdot, \cdot)$  be the self-similar metric on  $\partial\Gamma$  coming from the geometric action of  $\Gamma$  on  $\mathcal{DG}(\Sigma)$  (see Definition 3.2).

As a consequence of Proposition 7.19, a general result about hyperbolic spaces due to M. Gromov states that there exist two constants  $A \geq 1$  and  $\alpha > 0$  such that for any  $\xi, \xi' \in \partial\Gamma$ :

$$A^{-1} e^{-\alpha\{\xi|\xi'\}_{x_0}} \leq d(\xi, \xi') \leq A e^{-\alpha\{\xi|\xi'\}_{x_0}}.$$

In the sequel we also write

$$d(\xi, \xi') \asymp e^{-\alpha\{\xi|\xi'\}_{x_0}}.$$

This means that,  $d(\xi, \xi')$  is, up to a multiplicative constant, equal to  $e^{-\alpha\{\xi|\xi'\}_{x_0}}$ . An application of this description of the visual metric on  $\partial\Gamma$  is the following proposition.

**Proposition 7.20.** *For every  $\epsilon > 0$ , there exists only a finite set of parabolic limit sets of diameter larger than  $\epsilon$ .*

*Proof.* Let  $\partial P$  be a parabolic limit set. Let  $g'\Sigma_I$  be a residue in  $\Sigma$  such that  $\partial P \simeq \partial(g'\Sigma_I)$ . According to Proposition 7.1, there exists a unique chamber  $x \subset g'\Sigma_I$  such that for every chamber  $y \subset g'\Sigma_I$  there exists a minimal gallery from  $x_0$  to  $y$  passing through  $x$ . Let  $g \in \Gamma$  such that  $x = gx_0$ . Then the diameter of  $\partial P$  is controlled by  $e^{-\alpha|g|}$  with  $|g| = d_c(x_0, gx_0)$ . As there exists only a finite number of  $g \in \Gamma$  such that  $|g|$  is smaller than a fixed constant, the proposition is proved.  $\square$

## 7.4 Approximation of $\partial\Gamma$ with shadows

The following proposition says that shadows are almost balls. This will allow us to construct approximations using shadows.

**Proposition 7.21.** *There exists  $\lambda > 1$  such that for any  $x \in \text{Ch}(\Sigma)$  with  $d_c(x_0, x) = k$  there exists  $z \in v_x$  with*

$$B(z, \lambda^{-1}e^{-\alpha k}) \subset v_x \subset B(z, \lambda e^{-\alpha k}).$$

*Proof.* To prove the right-hand side inclusion it is enough to notice that  $\text{diam } v_x \leq Ae^{-\alpha k}$  where  $A$  and  $\alpha$  are the visual constants. Let  $C_x$  be the cone based at  $x$ . Let  $C = D_0(M_1) \cap \dots \cap D_0(M_k)$  and  $g \in \Gamma$  such that  $g(C) = C_x$  (see Proposition 7.10). Now we recall that  $g^{-1}$  is a bi-Lipschitz homeomorphism. Restricted to  $v_x$ , it rescales the metric by a factor  $e^{\alpha k}$ . According to Proposition 7.12, there exist  $r > 0$  and  $z \in \partial C$  such that  $B(z, r) \subset \partial C$ . As there is only a finite number of possible  $C$ , the proof is achieved.  $\square$

Let  $x \in \text{Ch}(\Sigma)$  and  $v_x$  be the associated shadow as in Definition 7.13. Thanks to Proposition 7.19, if  $d_c(x_0, x) = k$  then  $\text{diam } v_x \asymp e^{-\alpha k}$ . We use this property to construct an approximation of  $\partial\Gamma$  consisting of shadows.

For an integer  $k \geq 0$  we set

$$S_k = \{x \in \text{Ch}(\Sigma) : d_c(x_0, x) = k\}.$$

The set  $\{v_x : x \in S_k\}$  is a finite covering of  $\partial\Gamma$ . Now let  $S'_k$  be a subset of  $S_k$  such that  $\{v_x : x \in S'_k\}$  defines a minimal covering of  $\partial\Gamma$ . This means that for every  $x \in S'_k$  there exists  $z \in v_x$  such that  $z \notin v_y$  for any  $y \in S'_k \setminus \{x\}$ . Finally we set

$$G_k = \{v_x : x \in S'_k\}.$$

In the following, we prove that the sequence  $\{G_k\}_{k \geq 0}$  defines an approximation of  $\partial\Gamma$ .

**Proposition 7.22.** *For  $k \geq 0$ , let  $S'_k$  be the set of chambers previously defined and  $G_k$  be the minimal covering of  $\partial\Gamma$  associated with  $S'_k$ . There exists  $\kappa > 1$  such that for any  $x \in S'_k$ , there exists  $\xi_x \in v_x$  such that:*

- $\forall x \in S'_k: B(\xi_x, \kappa^{-1}e^{-\alpha k}) \subset v_x \subset B(\xi_x, \kappa e^{-\alpha k})$ ,
- $\forall x, y \in S'_k$  with  $x \neq y$ :  $B(\xi_x, \kappa^{-1}e^{-\alpha k}) \cap B(\xi_y, \kappa^{-1}e^{-\alpha k}) = \emptyset$ .

This property is enough to construct an approximation of  $\partial\Gamma$ . Indeed the visual constant  $\alpha$  can be chosen such that  $1/2 \leq e^{-\alpha} < 1$ . In this case we can extract from  $\{G_k\}_{k \geq 0}$  a subsequence that is an approximation of  $\partial\Gamma$  as defined in Subsection 2.1.

*Proof of Proposition 7.22.* Let  $x \in S'_k$ , and let  $\xi_x \in v_x$ . With  $\text{diam } v_x \asymp e^{-\alpha k}$ , there exists  $\kappa > 1$  such that for all  $x \in S'_k$ :  $v_x \subset B(\xi_x, \kappa e^{-\alpha k})$ .

We recall that the hyperbolicity provides a constant  $N \geq 1$ , depending only on the hyperbolicity parameter, such that for  $x, x' \in \text{Ch}(\Sigma)$  with  $d_c(x_0, x) = d_c(x_0, x')$  if  $d_c(x, x') \geq N$  then  $v_x \cap v_{x'} = \emptyset$ .

For any  $x \in S'_k$ , we pick  $z_x \in v_x$  such that  $z_x \notin v_y$  for any  $y \in S'_k \setminus \{x\}$ . Let  $x, y \in S'_k$ ,  $x \neq y$  and let  $c \in \text{Ch}(\Sigma)$  be such that  $d_c(x_0, c) = \{z_x | z_y\}_{x_0}$  and  $\{z_x, z_y\} \subset v_c$ . In this setting we can write that  $z_x$  and  $z_y$  are represented by infinite minimal galleries of the form:

- $z_x = [x_0 \sim x_1 \sim \dots \sim x_i \sim \dots \sim x_k \sim x_{k+1} \sim \dots]$
- $z_y = [x_0 \sim y_1 \sim \dots \sim y_i \sim \dots \sim y_k \sim y_{k+1} \sim \dots]$

with

- $x_i = c$  and  $y_i = c$  for one  $i \in \{1, \dots, k-1\}$ ,
- $x_k = x$  and  $y_k = y$ .

Now we consider two cases. First, we assume that  $x_{i+1}$  is adjacent to  $y_{i+1}$ . In this case, Proposition 5.23 implies that the bi-infinite gallery

$$\dots \sim x_{k+1} \sim x_k \sim \dots \sim x_{i+1} \sim y_{i+1} \sim \dots \sim y_k \sim y_{k+1} \sim \dots,$$

only crosses once the building-walls that separate  $\xi$  and  $\xi'$ . Hence it is minimal.

Then we assume that  $x_{i+1}$  is not adjacent to  $y_{i+1}$ . In this case we apply the same reasoning to the bi-infinite gallery

$$\dots \sim x_{k+1} \sim x_k \sim \dots \sim x_{i+1} \sim c \sim y_{i+1} \sim \dots \sim y_k \sim y_{k+1} \sim \dots.$$

Summarizing we obtain that one of the following galleries:

- $\dots \sim x_{k+1} \sim x_k \sim \dots \sim x_{i+1} \sim y_{i+1} \sim \dots \sim y_k \sim y_{k+1} \sim \dots$
- $\dots \sim x_{k+1} \sim x_k \sim \dots \sim x_{i+1} \sim c \sim y_{i+1} \sim \dots \sim y_k \sim y_{k+1} \sim \dots$

is a bi-infinite minimal gallery from  $z_x$  to  $z_y$ .

In particular  $d_c(x_{k+N}, y_{k+N}) > N$  and the corresponding shadows do not intersect:

$$v_{x_{k+N}} \cap v_{y_{k+N}} = \emptyset.$$

Now according to Proposition 7.21 there exist  $\xi_x \in v_{x_{k+N}}$  and  $\xi_y \in v_{y_{k+N}}$  such that

$$B(\xi_x, \lambda^{-1}e^{-\alpha(k+N)}) \subset v_{x_{k+N}} \text{ and } B(\xi_y, \lambda^{-1}e^{-\alpha(k+N)}) \subset v_{y_{k+N}}.$$

With  $v_{x_{k+N}} \cap v_{y_{k+N}} = \emptyset$ ,  $v_{x_{k+N}} \subset v_x$ , and  $v_{y_{k+N}} \subset v_y$  we obtain the desired property.  $\square$

## 8 Modulus in the boundary of a building and in the boundary of an apartment

The boundary of an apartment is, in a well chosen case, easier to understand than the boundary of the building. This is why we want to compare the modulus in the boundary of the building with some modulus in the boundary of an apartment.

In this section, we start by defining a convenient approximations on  $\partial\Gamma$  and on the boundaries of the apartments using shadows and retraction maps. Afterwards, we introduce the weighted modulus on the boundary of an apartment. Then we prove Theorem 8.9. This theorem is, after Theorem 6.12, the second major step in proving the main theorem (Theorem 10.1). It states that weighted modulus are comparable to the modulus in  $\partial\Gamma$ . Finally, using the ideas in Subsection 3.2, we reveal a connection between the conformal dimension of  $\partial\Gamma$  and a critical exponent computed in the boundary of an apartment.

We use the notation and assumptions from Sections 5, 6 and 7. In particular  $p \geq 1$  is a fixed constant. We fix  $\Gamma$  the graph product associated with the pair  $(\mathcal{G}, \{\mathbb{Z}/q_i\mathbb{Z}\}_{i=1,\dots,n})$ . The self-similar metric  $d(\cdot, \cdot)$  on  $\partial\Gamma$  is defined as in Subsection 7.3. The visual exponent of  $d(\cdot, \cdot)$  is  $\alpha$ . As in Section 3,  $d_0$  denotes a small constant compared with  $\text{diam } \partial\Gamma$  and with the constant of approximate self-similarity. Then  $\mathcal{F}_0$  is the set of curves of diameter larger than  $d_0$ .

### 8.1 Notations and conventions in $\partial A$ and in $\partial\Gamma$

In the rest of this article we fix an apartment  $A$  containing the base chamber  $x_0$ . We shall connect the geometry and the modulus in  $\partial A$  and in  $\partial\Gamma$ . Naturally we will use in  $\partial A$  and in  $\partial\Gamma$  the same concepts. Here we summarize some of the notation used in the following to avoid confusion. First we write

$$\mathcal{A}p_0(\Sigma) = \{B \in \mathcal{A}p(\Sigma) : x_0 \subset B\}.$$



Let  $\pi$  denote the retraction  $\pi_{A,x_0} : \Sigma \longrightarrow A$ . We also denote by  $\pi$  the extension of the retraction to the boundary. The notation  $d(\cdot, \cdot)$  and  $\alpha$  are also used to describe the metric on  $\partial B$  for any  $B \in \mathcal{A}p_0(\Sigma)$ .

An apartment is a thin building, so we can use in  $\partial A$  the tools presented in Subsections 7.2 and 7.3. First, we define on  $\partial A$  a combinatorial self-similar metric as in Subsection 7.3. Since  $x_0 \subset A$ , for  $\xi, \xi' \in \partial A$ , the quantity  $\{\xi|\xi'\}_{x_0}$  is the same whether we compute it in  $A$  or in  $\Sigma$ . Hence, if we choose the same visual exponents for the visual metric in  $\partial\Gamma$  and the visual metric in  $\partial A$ , then the metrics coincide up to a multiplicative constant. In the rest,  $d(\cdot, \cdot)$  designate the metric on both  $\partial A$  and  $\partial\Gamma$ . Likewise,  $\alpha$  and  $A$  designate the visual constants or both  $\partial A$  and  $\partial\Gamma$ .

Finally, it makes sense to talk about cones of chambers in  $A$  and shadows in  $\partial A$ . The results of 7.2 also hold in  $\partial A$ .

### Notation.

- for  $\xi \in \partial A$  and  $r > 0$  we designate by  $B(\xi, r) \subset \partial\Gamma$  the open ball of  $\partial\Gamma$  of radius  $r$  and center  $\xi$ ,
- for  $x \in \text{Ch}(A)$  we write  $C_x$  for the cone of chambers based on  $x$  in  $\Sigma$ ,
- for  $x \in \text{Ch}(A)$  we write  $v_x$  (resp.  $w_x$ ) for the shadow of  $x$  in  $\partial\Gamma$  (resp.  $\partial A$ ).

Usually we will use the following conventions.

- $v$  (resp.  $w$ ) designates an open subset of  $\partial\Gamma$  (resp. of  $\partial A$ ),
- $\partial P$  (resp.  $\partial Q$ ) designates a parabolic limit set in  $\partial\Gamma$  (resp. in  $\partial A$ ).

## 8.2 Choice of approximations

The following lemma says that shadows have a nice behavior under retraction maps.

**Lemma 8.1.** *Let  $A \in \mathcal{A}p_0(\Sigma)$  and let  $x \in \text{Ch}(\Sigma)$  and  $v_x$  be the associated shadow in  $\partial\Gamma$  as defined in Definition 7.13. Then*

- either  $x \notin \text{Ch}(A)$  and  $\text{Int}(v_x) \cap \partial A = \emptyset$ ,
- or  $x \in \text{Ch}(A)$  and  $v_x \cap \partial A$  is a shadow in  $\partial A$ .

*In the second case  $v_x \cap \partial A = \pi(v_x)$ .*

*Proof.* Let  $C_x$  be the cone based on  $x$ . If  $\text{Int}(v_x) \cap \partial A \neq \emptyset$  then there exists a chamber  $c$  in  $A \cap C_x$ . By convexity, a minimal gallery from  $x_0$  to  $c$  that passes through  $x$  is included in  $A$  and  $x \subset A$ . Therefore  $v_x \cap \partial A$  is the shadow in  $\partial A$  associated with  $x$ .  $\square$

We fix  $\{G_k^A\}_{k \geq 0}$  an approximation of  $\partial A$  based on shadows as constructed in Subsection 7.4.

**Notation.** For  $k \geq 0$  we set

$$G_k := \{v_y \subset \partial \Gamma : \pi(v_y) \in G_k^A\}.$$

We recall that we chose the same visual exponents for the metrics in  $\partial \Gamma$  and  $\partial A$ . As a consequence of Lemma 8.1 we get the following fact.

**Fact 8.2.** *There exists  $\kappa > 1$  such that  $\{G_k^A\}_{k \geq 0}$  and  $\{G_k\}_{k \geq 0}$  are  $\kappa$ -approximations. Moreover, for any  $w \in G_k^A$  there exists a unique  $\tilde{w} \in G_k$  such that  $\text{Int}(\tilde{w}) \cap \partial A \neq \emptyset$  and  $\pi(\tilde{w}) = w$ .*

Hereafter,  $\{G_k\}_{k \geq 0}$  designates the approximation of  $\partial \Gamma$  obtained from  $\{G_k^A\}_{k \geq 0}$  thanks to the preceding fact. This approximation of  $\partial \Gamma$  is canonically associated with  $\{G_k^A\}_{k \geq 0}$  in the following sense: from  $\{G_k\}_{k \geq 0}$  we can equip any  $B \in \mathcal{A}p_0(\Sigma)$  with an approximation isometric to  $\{G_k^A\}_{k \geq 0}$ . Indeed if  $B \in \mathcal{A}p_0(\Sigma)$ , for  $k \geq 0$  we set

$$G_k^B := \{w = \partial B \cap v : v \in G_k\}.$$

Now let  $B \in \mathcal{A}p_0(\Sigma)$  and  $f : B \rightarrow A$  be the type preserving isometry that fixes  $x_0$ . The map  $f$  is realized by the restriction to  $B$  of the retraction  $\pi$  and we get the following fact.

**Fact 8.3.**  $G_k^A = \{f(v)\}_{v \in G_k^B}$ .

Now that an approximation  $\{G_k\}_{k \geq 0}$  is fixed the results we will obtain on the combinatorial modulus in  $\partial \Gamma$  will be valid, up to multiplicative constants, for any approximation thanks to Proposition 2.5.

### 8.3 Weighted modulus in $\partial A$

On scale  $k \geq 0$ , to compare the modulus in the building with the modulus in the apartment we need to compare the cardinality of  $G_k$  with the cardinality of  $G_k^A$ . If the building is thick these quantities differ by an exponential factor in  $k$ . This is the reason we attach a weight to the elements of  $G_k^A$ .

**Definition 8.4.** Let  $w \in G_k^A$ , we set  $q(w) = \#\{v \in G_k : \pi(v) = w\}$ .

Let  $k \geq 0$  and let  $\mathcal{F}^A$  be a set of curves contained in  $\partial A$ . As in Subsection 2.1, a positive function  $\rho : G_k^A \rightarrow [0, +\infty)$  is said to be  $\mathcal{F}^A$ -admissible if for any  $\gamma \in \mathcal{F}^A$

$$\sum_{\gamma \cap w \neq \emptyset} \rho(w) \geq 1.$$

The *weighted  $p$ -mass* of  $\rho$  in  $\partial A$  is

$$WM_p^A(\rho) = \sum_{w \in G_k^A} q(w) \rho(w)^p.$$

**Definition 8.5.** Let  $k \geq 0$  and let  $\mathcal{F}^A$  be a set of curves contained in  $\partial A$ , we define the weighted  $G_k^A$ -combinatorial  $p$ -modulus of  $\mathcal{F}^A$  by

$$\text{Mod}_p^A(\mathcal{F}^A, G_k^A) := \inf\{WM_p^A(\rho)\},$$

where the infimum is taken over the set of  $\mathcal{F}^A$ -admissible functions and with the convention  $\text{Mod}_p^A(\emptyset, G_k^A) = 0$ . For simplicity, we usually use the terminology *weighted modulus*.

We can check that Proposition 2.3 holds for weighted modulus as well and the proof is identical to the one for the usual combinatorial modulus.

This definition of the weighted modulus strongly depends on the choice we have made for the approximation. In particular, it does not permit to compute a weighted modulus relatively to a generic approximation of  $\partial A$ . As a consequence, an analogue to Proposition 2.5 would make no sense here. This is a huge restriction on the use of the weighted modulus. Indeed, this proposition is essential in proving Proposition 3.11 and Theorem 6.12 for the usual combinatorial modulus.

However, in the rest of the paper, the weighted modulus will be used to compute inequalities for the usual combinatorial modulus. As the usual combinatorial modulus, up to multiplicative constant, does not depend on the choice of the approximation, this point will not be a problem for us.

The following proposition says that the weights are given by the types of the building-walls crossed by a minimal gallery. We recall that the group  $\Gamma$  is given by the following presentation

$$\Gamma = \langle s_i \in S \mid s_i^{q_i} = 1, s_i s_j = s_j s_i \text{ if } v_i \sim v_j \rangle.$$

**Proposition 8.6.** Let  $w \in G_k^A$  be such that  $w$  is a shadow  $w = w_x$  for  $x \in \text{Ch}(A)$ . Let  $x_0 \sim_{s_1} x_1 \sim_{s_2} \cdots \sim_{s_{k-1}} x_{k-1} \sim_{s_k} x$  be a minimal gallery where  $s_i$  is the generator of  $\Gamma$  associated with the type of the building-wall between  $x_{i-1}$  and  $x_i$  for every  $1 \leq i \leq k$ . If  $q_i$  is the order of  $s_i$  for every  $1 \leq i \leq k$  then

$$q(w) = \prod_{i=1, \dots, k} q_i - 1.$$

*Proof.* Let  $w \in G_k^A$  and  $x \in \text{Ch}(A)$  be such that  $w = w_x$  in  $\partial A$ . Then we observe that  $\{v \subset \partial \Gamma : \pi(v) = w\} = \{v_y \subset \partial \Gamma : \pi(y) = x\}$ . As a consequence, we obtain  $q(w) = \#\pi^{-1}(x)$ .

Now consider the gallery  $x_0 \sim_{s_1} x_1 \sim_{s_2} \cdots \sim_{s_{k-1}} x_{k-1} \sim_{s_k} x$  given in the statement of proposition. Since  $\pi$  preserves the types,  $y \in \text{Ch}(\Sigma)$  is in  $\pi^{-1}(x)$  if and only if there exists a

minimal gallery from  $x_0$  to  $y$  in  $\Sigma$  of the form  $x_0 \sim_{s_1} y_1 \sim_{s_2} \cdots \sim_{s_{k-1}} y_{k-1} \sim_{s_k} y$ . Finally, we obtain  $q(w) = \prod_{i=1, \dots, k} q_i - 1$ .  $\square$

Thanks to the choices we have made, the weighted modulus is invariant up to a change of apartment in the following sense. For  $B \in \mathcal{A}p_0(\Sigma)$  consider the approximation  $G_k^B$  given by Fact 8.3. To any element  $w \in G_k^B$  we attach a weight and define a *weighted  $G_k^B$ -combinatorial  $p$ -modulus* as it is done in  $\partial A$ . Now let  $f : B \rightarrow A$  be a type preserving isometry that fixes  $x_0$  and denote  $f : \partial B \rightarrow \partial A$  the extension of this map to the boundary. The map  $f$  is realized by the restriction of the retraction  $\pi$  to  $B$ . Thus  $f$  preserves the weights. Then the following fact is an immediate consequence of Fact 8.3.

**Fact 8.7.** *Let  $B \in \mathcal{A}p_0(\Sigma)$ . Then for any  $k \geq 0$  and any set of curves  $\mathcal{F}^B$  contained in  $\partial B$  one has*

$$\text{Mod}_p^B(\mathcal{F}^B, G_k^B) = \text{Mod}_p^A(f(\mathcal{F}^B), G_k^A).$$

Note that, for any  $k \geq 0$  and any  $w \in G_k^A$  one has

$$1 \leq q(w) \leq (q-1)^k \text{ with } q := \max\{q_1, \dots, q_n\}.$$

Therefore for any set of curves  $\mathcal{F}^A$  contained in  $\partial A$ , the next inequalities follow directly from the definition

$$\text{mod}_p^A(\mathcal{F}^A, G_k^A) \leq \text{Mod}_p^A(\mathcal{F}^A, G_k^A) \leq (q-1)^k \text{mod}_p^A(\mathcal{F}^A, G_k^A),$$

where the modulus in small letters designates the usual *modulus computed in  $\partial A$* . In particular if  $\Gamma$  is of constant thickness  $q \geq 3$  then

$$\text{Mod}_p^A(\mathcal{F}^A, G_k^A) = (q-1)^k \text{mod}_p^A(\mathcal{F}^A, G_k^A).$$

As a consequence, at fixed scale  $k \geq 0$ , the weighted modulus depends only on the boundary of an apartment. We will discuss this particular case in Sections 9 and 10.

The following proposition is a major motivation of the definition of the weighted modulus.

**Proposition 8.8.** *Let  $\mathcal{F}$  be a set of curves in  $\partial \Gamma$  and let  $\mathcal{F}^A$  be a set of curves in  $\partial A$  such that  $\pi(\mathcal{F}) \subset \mathcal{F}^A$ . Then*

$$\text{Mod}_p(\mathcal{F}, G_k) \leq \text{Mod}_p^A(\mathcal{F}^A, G_k^A).$$

*Proof.* Let  $\rho_A$  be a  $\mathcal{F}^A$ -admissible function. We set  $\rho : G_k \rightarrow [0, +\infty)$  defined by

$$\rho(v) = \rho_A \circ \pi(v).$$

If  $\gamma \in \mathcal{F}$ , let  $\gamma_A := \pi \circ \gamma$ . Then, as  $\gamma_A \in \mathcal{F}^A$

$$L_\rho(\gamma) = \sum_{v \cap \gamma \neq \emptyset} \rho_A \circ \pi(v) \geq \sum_{w \cap \gamma_A \neq \emptyset} \rho_A(w) \geq 1,$$

thus  $\rho$  is  $\mathcal{F}$ -admissible. Furthermore, one has:

$$M_p(\rho) = \sum_{v \in G_k} \rho_A \circ \pi(v)^p = \sum_{w \in G_k^A} q(w) \cdot \rho_A(w)^p = WM_p^A(\rho_A).$$

With the first point it follows that  $\text{Mod}_p(\mathcal{F}, G_k) \leq \text{Mod}_p^A(\mathcal{F}^A, G_k^A)$ .  $\square$

## 8.4 Modulus in $\partial\Gamma$ compared with weighted modulus in $\partial A$

As before  $d_0 > 0$  is a small constant compared with  $\text{diam } \partial\Gamma$  and with the constant of approximate self-similarity.

We recall that the apartment  $A \in \mathcal{A}p_0(\Sigma)$  is fixed. Thanks to Fact 8.7 the following result hold for any apartment containing  $x_0$ .

In this subsection we continue to use the approximations  $G_k$  and  $G_k^A$  defined at the beginning of Subsection 8.2. We recall that if  $\eta$  is a non-constant curve of  $\partial\Gamma$ , the notation  $\mathcal{U}_\epsilon(\eta)$  designates the  $\epsilon$ -neighborhood of  $\eta$  relative to the  $C^0$  topology. If  $\eta$  is a non-constant curve contained in  $\partial A$ , we use the notation

$$\mathcal{U}_\epsilon^A(\eta) := \{\gamma \in \mathcal{U}_\epsilon(\eta) : \gamma \subset \partial A\}.$$

The next theorem says that in this case, the modulus of  $\mathcal{U}_\epsilon(\eta)$  in the boundary of the building is controlled by the weighted modulus of  $\mathcal{U}_\epsilon^A(\eta)$  in the boundary of the apartment. It is a key step in the proof of Theorem 10.1.

**Theorem 8.9.** *Let  $p \geq 1$ , let  $\eta \in \mathcal{F}_0$  and assume  $\eta \subset \partial A$ . For  $\epsilon > 0$  small enough so that the hypothesis of Theorem 6.12 hold in  $\partial\Gamma$ , there exists a positive constant  $C = C(p, \eta, \epsilon)$  independent of  $k$  such that for every  $k \geq 1$  large enough*

$$\text{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k) \leq \text{Mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A) \leq C \cdot \text{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

Furthermore, when  $p$  belongs to a compact subset of  $[1, +\infty)$  the constant  $C$  may be chosen independent of  $p$ .

For the rest of the subsection  $\eta \in \mathcal{F}_0$  and  $\epsilon > 0$  are as in the hypothesis of the preceding theorem. For each  $\eta \in \mathcal{F}_0$  we fix a constant  $r > 0$  such that the hypothesis of Theorem 6.12 are satisfied. To prove the theorem we need to introduce the following notation:

- $\text{Aut}_\Sigma$  is the full group of type preserving isometries of  $\Sigma$ .
- For  $n \geq 0$ ,  $B_n \subset \text{Ch}(\Sigma)$  is the ball of center  $x_0$  and of radius  $n$  for the distance over the chambers  $d_c(\cdot, \cdot)$ .
- For  $n \geq 0$ ,  $K_n < \text{Aut}_\Sigma$  is the pointwise stabilizer of  $B_n$  under the action of  $\text{Aut}_\Sigma$ .
- $\mathcal{F}_n := \{g\gamma \subset \partial\Gamma : g \in K_n \text{ and } \gamma \in \mathcal{U}_\epsilon^A(\eta)\}$ .

The main step to prove the theorem is to show that  $\mathcal{F}_n$  is an intermediate set of curves between  $\mathcal{U}_\epsilon^A(\eta)$  and  $\mathcal{U}_\epsilon(\eta)$ . This will be done in Lemma 8.11. Before proving this, we need to discuss the action of  $K_n$  on the chambers. The next lemma uses ideas of [Cap14, Lemma 3.5 and Proposition 8.1].

**Lemma 8.10.** *There exists an integer  $N > 0$  depending only on  $n$  and satisfying the following property. Let  $x \in \text{Ch}(\Sigma)$ , set  $d_c(x_0, x) = k$  and assume  $k > n$ . Let*

$$x_0 \sim_{s_1} x_1 \sim_{s_2} \cdots \sim_{s_n} x_n \sim_{s_{n+1}} \cdots \sim_{s_{k-1}} x_{k-1} \sim_{s_k} x$$

*be a minimal gallery where  $s_1, \dots, s_k$  is the family of types of the building-walls crossed by this gallery. Then*

$$\frac{1}{(q-1)^N} \cdot \prod_{i=n+1}^k q_i - 1 \leq \#K_n.x \leq \prod_{i=n+1}^k q_i - 1,$$

where  $q := \max\{q_1, \dots, q_n\}$ .

*Proof.* Since  $K_n$  preserves the types and fixes  $x_0, \dots, x_n$  it follows that

$$\#K_n.x \leq \prod_{i=n+1}^k q_i - 1.$$

Now for  $D \in \mathcal{D}(\Sigma)$  we write  $U(D)$  the pointwise stabilizer of  $D$  under the action of  $\text{Aut}_\Sigma$  and we set

$$U(n) = \langle U(D) | B_n \subset \text{Ch}(D) \rangle.$$

Clearly  $U(n) < K_n$  and

$$\#K_n.x \geq \#U(n).x.$$

Now if we write  $M_i$  the building-wall between  $x_i$  and  $x_{i+1}$ , we observe that the orbit of  $x_{i+1}$  under  $U(D_0(M_i))$  has  $q_i - 1$  elements. Indeed,  $U(D_0(M_i))$  acts as the full group of permutations on the set  $\{D_1(M_i), \dots, D_{q_i-1}(M_i)\}$ .

Note that  $U(D_0(M_i)) < U(n)$  if and only if  $B_n \subset \text{Ch}(D_0(M_i))$ . Otherwise  $M_i$  crosses  $B_n$ , because  $x_0 \in \text{Ch}(D_0(M_i))$ . As a consequence, we set  $N$  the number of building-walls that cross  $B_n$  and we obtain

$$\#U(n).x \geq \frac{1}{(q-1)^N} \cdot \prod_{i=n+1}^k q_i - 1.$$

This achieves the proof. □

Now we can prove the main lemma.

**Lemma 8.11.** *Let  $p \geq 1$ . For  $n \geq 0$  large enough, there exist two positive constants  $C_1, C_2$  depending only on  $p, \eta, \epsilon$ , such that for every  $k > n$ :*

$$\text{Mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A) \leq C_1 \cdot \text{Mod}_p(\mathcal{F}_n, G_k) \leq C_2 \cdot \text{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

*Furthermore, when  $p$  belongs to a compact subset of  $[1, +\infty)$  the constants may be chosen independent of  $p$ .*

*Proof.* i) First we prove the right-hand side inequality. According to Proposition 7.19, for any  $g \in K_n$  and any  $\xi \in \partial\Gamma$ ,  $d(\xi, g\xi) \leq A.e^{-\alpha n}$  where  $A$  is the visual multiplicative constant. Hence for  $n \geq 0$  large enough, by triangular inequality,  $\mathcal{F}_n \subset \mathcal{U}_{2\epsilon}(\eta)$ . Then, as a consequence of Theorem 6.12, for a fixed  $r > 0$  the combinatorial modulus of  $\mathcal{U}_\epsilon(\eta)$  is controlled by the combinatorial modulus of  $\mathcal{U}_{2\epsilon}(\eta)$  with multiplicative constants depending only on  $p, \eta, \epsilon$ . Thus, by Proposition 2.3 (1), there exists  $C = C(p, \eta, \epsilon)$  such that

$$\text{Mod}_p(\mathcal{F}_n, G_k) \leq C \cdot \text{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

ii) Now we fix an integer  $n \geq 0$  large enough so that the first part of the proof holds. We use the notation  $K := K_n$  for simplicity. Moreover we assume that  $k > n$ . Let  $\rho : G_k \rightarrow [0, +\infty)$  be a minimal  $\mathcal{F}_n$ -admissible function and set  $\rho_A : G_k^A \rightarrow [0, +\infty)$  the function defined by:

$$\rho_A(w) = \int_K \rho(g\tilde{w})d\mu(g),$$

where  $\mu$  denotes the Haar probability measure over  $K$  and where the function  $w \in G_k^A \rightarrow \tilde{w} \in G_k$  is given by Fact 8.2. Let  $w \in G_k^A$  and let  $x \in \text{Ch}(\Sigma)$  be such that  $v_x = \tilde{w}$ . Then  $d_c(x_0, x) = k$ . As in Proposition 8.6, let

$$x_0 \sim_{s_1} x_1 \sim_{s_2} \cdots \sim_{s_n} x_n \sim_{s_{n+1}} \cdots \sim_{s_{k-1}} x_{k-1} \sim_{s_k} x$$

be a minimal gallery where  $s_1, \dots, s_k$  is the family of types of the building-walls crossed by this gallery. We set

$$q(w, n) = \prod_{i=n+1, \dots, k} q_i - 1.$$

We notice that for any  $g \in K$  the translated  $g\tilde{w} = gv_x$  is the shadow  $v_{gx}$ . In particular, this means that  $\#K.\tilde{w} = \#K.x$ . Then according to Lemma 8.10

$$(*) \quad \frac{q(w, n)}{(q-1)^N} \leq \#K.\tilde{w} \leq q(w, n),$$

where  $q := \max\{q_1, \dots, q_n\}$  and  $N$  is the number of building-walls crossing  $B_n$ .

As a consequence we can write

$$\rho_A(w) = \frac{1}{\#K.\tilde{w}} \cdot \sum_{v \in K.\tilde{w}} \rho(v),$$

and we prove the second inequality of the proposition.

On the one hand, let  $\gamma \in \mathcal{U}_\epsilon^A(\eta)$ :

$$L_{\rho_A}(\gamma) = \sum_{w \cap \gamma \neq \emptyset} \int_K \rho(g\tilde{w}) d\mu(g) = \int_K \sum_{w \cap \gamma \neq \emptyset} \rho(g\tilde{w}) d\mu(g) = \int_K \sum_{v \cap g(\gamma) \neq \emptyset} \rho(v) d\mu(g).$$

Since  $g(\gamma) \in \mathcal{F}_n$ , we get  $\sum_{v \cap g(\gamma) \neq \emptyset} \rho(v) \geq 1$  and  $\rho_A$  is  $\mathcal{F}_A$ -admissible.

On the other hand, thanks to Jensen's inequality, for  $p \geq 1$  one has:

$$WM_p^A(\rho_A) \leq \sum_{w \in G_k^A} q(w) \int_K \rho(g\tilde{w})^p d\mu(g) = \sum_{w \in G_k^A} \frac{q(w)}{\#K \cdot \tilde{w}} \cdot \sum_{v \in K \cdot \tilde{w}} \rho(v)^p.$$

Hence with (\*) we obtain

$$WM_p^A(\rho_A) \leq \sum_{w \in G_k^A} (q-1)^N \cdot \frac{q(w)}{q(w, n)} \cdot \sum_{v \in K \cdot \tilde{w}} \rho(v)^p \leq (q-1)^{n+N} M_p(\rho).$$

Finally we get:

$$\text{Mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A) \leq (q-1)^{n+N} \text{Mod}_p(\mathcal{F}_n, G_k).$$

This last multiplicative constant depends only on  $n$  and on the geometry of the building. Since  $n$  depends only on  $\eta$  and  $\epsilon$  the second inequality is proved.

iii) The last statement of the lemma is an immediate consequence of the two first parts and of Theorem 6.12. □

*Proof of Theorem 8.9.* Since  $\pi(\mathcal{U}_\epsilon(\eta)) \subset \mathcal{U}_\epsilon^A(\eta)$ , Proposition 8.8 and Lemma 8.11 imply the theorem. □

## 8.5 Consequences

Here we continue to use the approximations  $G_k$  and  $G_k^A$  defined in Subsection 8.2. For  $\eta$  a non-constant curve in  $\partial A$ ,  $\partial Q$  a parabolic limit set in  $\partial A$ , and  $\delta, r, \epsilon > 0$ , we use the following notation:

- $\mathcal{F}_0^A = \{\gamma \in \mathcal{F}_0 : \gamma \subset \partial A\}$ ,
- $\mathcal{F}_{\delta, r}^A(\partial Q)$  is the subset of  $\mathcal{F}_0^A$  consisting of all curves  $\gamma$  satisfying:
  - $\gamma \subset N_\delta(\partial Q)$ ,
  - $\gamma \not\subset N_r(\partial Q')$  for any connected parabolic limit set  $\partial Q' \subsetneq \partial Q$ ,



- $\delta_0(\cdot)$  refers to the increasing function in Theorem 6.12.

We recall that the apartment  $A \in \mathcal{A}p_0(\Sigma)$  is fixed. Thanks to Fact 8.7, the following result holds for any apartment containing  $x_0$ .

The main consequence of the previous subsection is Theorem 8.13, where we control from above the combinatorial modulus of  $\mathcal{F}_0$  by the weighted-combinatorial modulus of  $\mathcal{F}_0^A$ .

**Lemma 8.12.** *Let  $p \geq 1$  and  $A \in \mathcal{A}p_0(\Sigma)$ . Let  $\partial P$  be a parabolic limit set in  $\partial\Gamma$  and assume that  $x_0 \subset \text{Conv}(\partial P)$ . Let  $\gamma$  be a non-constant curve in  $\partial Q = \partial P \cap \partial A$  such that  $\partial Q$  is the smallest parabolic limit set of  $\partial A$  containing  $\gamma$ . Let  $r > 0$  be small enough so that  $\gamma \not\subset \overline{N_r}(\partial Q')$  for any connected parabolic limit set  $\partial Q' \subsetneq \partial Q$ . Let  $\delta < \delta_0(r)$ . Then for  $\epsilon > 0$  small enough, there exists a constant  $C = C(p, \gamma, r, \epsilon)$  such that for every  $k \geq 1$*

$$\text{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G_k) \leq C \cdot \text{Mod}_p(\mathcal{U}_\epsilon(\gamma), G_k) \leq C \cdot \text{Mod}_p^A(\mathcal{F}_{\delta,r}^A(\partial Q), G_k^A).$$

In particular

$$\text{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G_k) \leq C \cdot \text{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

Furthermore, when  $p$  belongs to a compact subset of  $[1, +\infty)$  the constant  $C$  may be chosen independent of  $p$ .

*Proof.* As in the beginning of the proof of Theorem 6.12 we can assume, without loss of generality, that for  $\epsilon > 0$  small enough

- $x_0 \subset \text{Conv}(\gamma)$  for every  $\gamma \in \mathcal{F}_0$ ,
- $\mathcal{U}_\epsilon(\gamma) \subset \mathcal{F}_0$ .

As before, the multiplicative constants resulting from these assumptions only depends on  $d_0$ .

Now for  $\epsilon > 0$  small enough, we obtain by Proposition 2.3(1) and the preceding assumption

$$\text{Mod}_p^A(\mathcal{U}_\epsilon^A(\gamma), G_k^A) \leq \text{Mod}_p^A(\mathcal{F}_{\delta,r}^A(\partial Q), G_k^A) \leq \text{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

Since  $\pi(\mathcal{U}_\epsilon(\gamma)) \subset \mathcal{U}_\epsilon^A(\gamma)$ , with Proposition 8.8 one has

$$\text{Mod}_p(\mathcal{U}_\epsilon(\gamma), G_k) \leq \text{Mod}_p^A(\mathcal{U}_\epsilon^A(\gamma), G_k^A).$$

Finally thanks to Theorem 6.12 there exists  $C = C(p, \gamma, r, \epsilon)$  such that for every  $k \geq 1$

$$\text{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G_k) \leq C \cdot \text{Mod}_p(\mathcal{U}_\epsilon(\gamma), G_k).$$

□

Now we are ready to prove the following theorem which is used in the proof of Theorem 10.1.

**Theorem 8.13.** *For any  $p \geq 1$ , there exists a constant  $D = D(p)$  such that for every  $k \geq 1$*

$$\text{Mod}_p(\mathcal{F}_0, G_k) \leq D \cdot \text{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

*Proof.* i) First, we recall the following notation. For  $\delta, r > 0$  and for  $\partial P$  a connected parabolic limit set,  $\mathcal{F}_{\delta, r}(\partial P)$  is the set of all the curves in  $\mathcal{F}_0$  satisfying:

- $\gamma \subset N_\delta(\partial P)$ ,
- $\gamma \not\subset N_r(\partial P')$  for any connected parabolic limit set  $\partial P' \subsetneq \partial P$ .

Now, as it is done in [BK13, as a remark of Corollary 6.2.] in boundaries of Coxeter groups, we observe that  $\mathcal{F}_0$  splits into a finite union

$$\mathcal{F}_0 = \mathcal{F}_{\delta_1, r_1}(\partial P_1) \cup \dots \cup \mathcal{F}_{\delta_N, r_N}(\partial P_N)$$

with  $\delta_i < \delta_0(r_i)$ .

Indeed, let  $\mathcal{P} = \{\partial P_1, \dots, \partial P_N\}$  be the finite set (see Proposition 7.20) of all the parabolic limit sets of diameter larger than  $d_0$ . For  $\partial P \in \mathcal{P}$  we call *height* of  $\partial P$  the maximal length of a sequence in  $\mathcal{P}$  of the form

$$\partial P'_0 \subsetneq \partial P'_1 \subsetneq \dots \subsetneq \partial P'_i = \partial P.$$

Now, we index  $\mathcal{P}$  thanks to the height

$$\mathcal{P} = \{\partial P_{0,1}, \dots, \partial P_{0,N_0}, \dots, \partial P_{i,j}, \dots, \partial P_{M,1}, \dots, \partial P_{M,N_M}\}$$

where  $i$  is the height of  $\partial P_{i,j}$ . We fix a small  $r_0 > 0$  and  $\delta_0 < \delta_0(r_0)$ . Then by induction on the height we set

$$r_{i+1} = \delta_i \text{ and } \delta_{i+1} < \min\{\delta_0(r_{i+1}), \delta_i\}.$$

Now let  $\partial P \in \mathcal{P}$  be of height  $i > 0$ . By construction, for any  $i' < i$  we have  $r_i < \delta_{i'}$ . Hence for any  $\partial P' \subsetneq \partial P$  of height  $i'$  we have  $N_{r_i}(\partial P') \subset N_{\delta_{i'}}(\partial P')$  and

$$\mathcal{F}_0 = \bigcup_{i=0}^M \bigcup_{j=1}^{N_i} \mathcal{F}_{\delta_i, r_i}(\partial P_{i,j}).$$

ii) Let  $\partial P$  be one of the parabolic limit sets appearing in the preceding decomposition of  $\mathcal{F}_0$  and  $\delta, r > 0$  be the corresponding constants. As in the beginning of the proof of Theorem 6.12, we can assume that  $x_0 \subset \text{Conv}(\partial P)$ . Again the multiplicative constant resulting from this assumption only depends on  $d_0$ . Now pick  $B \in \mathcal{A}_{p_0}(\Sigma)$  such that  $\partial B \cap \partial P \neq \emptyset$  and fix

a curve  $\gamma$  and  $\epsilon > 0$  so that the hypothesis of Lemma 8.12 are satisfied. Then there exists  $C = C(p, \gamma, r, \epsilon)$  such that for every  $k \geq 1$

$$\text{Mod}_p(\mathcal{F}_{\delta, r}(\partial P), G_k) \leq C \cdot \text{Mod}_p^B(\mathcal{F}_0^B, G_0^B).$$

iii) With Fact 8.7, we observe that the weighted modulus on the right-hand side of the preceding inequality is independent of the choice of  $B \in \mathcal{A}p_0(\Sigma)$ . Finally, by Proposition 2.3 (2) and the two first parts of this proof, there exists a constant  $D = D(p)$  such that such that for every  $k \geq 1$

$$\text{Mod}_p(\mathcal{F}_0, G_k) \leq D \cdot \text{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

□

Note that for the moment we cannot prove a converse inequality between the modulus. Indeed, in the proof of Lemma 8.12 the use of Theorem 6.12 is a key point. As we said before, we cannot prove an analogue of Theorem 6.12 for the weighted modulus.

Nevertheless, we can define a critical exponent in connection with the weighted modulus as it is done in Subsection 3.2. Then Theorem 8.13 can be used to helps us understand this new critical exponent.

**Proposition 8.14.** *There exists  $p_0 \geq 1$  such that for  $p \geq p_0$  the weighted modulus  $\text{Mod}_p^A(\mathcal{F}_0^A, G_k^A)$  goes to zero as  $k$  goes to infinity.*

*Proof.* This proof is the same as the proof of Proposition 3.4. We recall that  $\kappa$  is the constant of the approximations  $\{G_k\}_{k \geq 0}$  and  $\{G_k^A\}_{k \geq 0}$ .

According to the doubling condition and the definition of an approximation, there exists an integer  $N'$  such that each element  $w \in G_k^A$  is covered by at most  $N'$  elements of  $G_{k+1}^A$ . As a consequence, if  $K > 0$  is the cardinality  $G_0$ , then

$$\#G_k^A \leq K \cdot N'^k \text{ for any } k \geq 1.$$

Moreover, as we saw in the proof of Proposition 2.4, there exists a constant  $K' > 0$  such that the constant function  $\rho : v \in G_k^A \longrightarrow \rho(v) = K' \cdot 2^{-k} \in [0, +\infty)$  is  $\mathcal{F}_0^A$ -admissible.

As a consequence

$$\text{mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq C \cdot \left(\frac{N'}{2^p}\right)^k,$$

where  $C$  is a positive constant. Hence we obtain

$$\text{Mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq (q-1)^k \cdot \text{mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq C \cdot \left(\frac{(q-1)N'}{2^p}\right)^k,$$

Thus, for  $p$  large enough,  $\text{Mod}_p^A(\mathcal{F}_0^A, G_k^A)$  goes to zero. □

It is now natural to define a critical exponent for the weighted modulus in the apartment.

**Definition 8.15.** *The critical exponent  $Q_W$  of the weighted modulus in  $\partial A$  is defined as follows*

$$Q_W = \inf\{p \in [1, +\infty) : \lim_{k \rightarrow +\infty} \text{Mod}_p^A(\mathcal{F}_0^A, G_k^A) = 0\}.$$

To avoid confusion, we use the following notation

- $Q$  for the critical exponent associated with the usual modulus  $\text{Mod}_p(\cdot, G_k)$  in  $\partial\Gamma$ ,
- $Q_A$  for the critical exponent associated with the usual modulus  $\text{mod}_p^A(\cdot, G_k^A)$  in  $\partial A$ ,
- $Q_W$  for the critical exponent associated with the weighted modulus  $\text{Mod}_p^A(\cdot, G_k^A)$  in  $\partial A$ .

We recall that  $Q$  and  $Q_A$  are respectively the conformal dimension of  $\partial\Gamma$  and of  $\partial A \simeq \partial W$  (see Theorem 3.7). The inequalities between the different modulus imply the following corollary.

**Corollary 8.16.** *The following inequalities hold*

$$Q_A \leq Q \leq Q_W.$$

*Proof.* With Proposition 2.3 (1) and Theorem 8.13, one has

$$\text{mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq \text{Mod}_p(\mathcal{F}_0, G_k) \leq D \cdot \text{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

The inequalities between the critical exponents follow. □

## 9 Application to buildings of constant thickness

In this section we use the notation and the assumptions from the previous section. In particular, the self-similar metric on  $\partial\Gamma$  is  $d(\cdot, \cdot)$ . We fix  $d_0$  a small constant compared with  $\text{diam } \partial\Gamma$  and with the constant of approximate self-similarity. Then  $\mathcal{F}_0$  is the set of curves of diameter larger than  $d_0$ . The notation  $\delta_0(\cdot)$  still refers to the increasing function in Theorem 6.12.

As before we fix an apartment  $A \in \mathcal{A}p_0(\Sigma)$  and  $\mathcal{F}_0^A$  is the set of curves in  $\partial A$  of diameter larger than  $d_0$ .

We assume that  $\Sigma$  is of constant thickness  $q \geq 3$ . This means that  $\Gamma$  is the graph product given by the pair  $(\mathcal{G}, \{\mathbb{Z}/q\mathbb{Z}\}_{i=1, \dots, n})$ . As before  $\{G_k^A\}_{k \geq 0}$  and  $\{G_k\}_{k \geq 0}$  are the approximations of  $\partial A$  and  $\partial\Gamma$  provided by Fact 8.2. We already noticed that, with the constant thickness assumption, we obtain for  $k \geq 0$  and  $\mathcal{F}^A$  a set of curves contained in  $\partial A$

$$\text{Mod}_p^A(\mathcal{F}^A, G_k^A) = q^k \text{mod}_p^A(\mathcal{F}^A, G_k^A),$$

where the modulus in small letters designates the usual modulus computed in  $\partial A$ . In particular, this means that from Theorem 6.12 applied to  $\text{mod}_p^A(\cdot, G_k^A)$  we can obtain analogous inequalities for  $\text{Mod}_p^A(\cdot, G_k^A)$ .

The constant thickness allows us to control by below the combinatorial modulus of  $\mathcal{F}_0$  by the weighted-combinatorial modulus of  $\mathcal{F}_0^A$ . Combining with the results of Subsection 8.5, we obtain a full control of the two modulus.

**Theorem 9.1.** *For any  $p \geq 1$ , there exists a constant  $D = D(p)$  such that for every  $k \geq 1$*

$$D^{-1} \cdot \text{Mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq \text{Mod}_p(\mathcal{F}_0, G_k) \leq D \cdot \text{Mod}_p^A(\mathcal{F}_0^A, G_k^A).$$

*In particular  $Q_W = Q$ .*

*Proof.* The right-hand side inequality is given by Theorem 8.13. The proof is almost the same for the left-hand side inequality. Indeed,  $\mathcal{F}_0^A$  admits a decomposition analogous to the decomposition used at the beginning of the proof of Theorem 8.13. With a fixed such decomposition and with Proposition 2.3 (2), it is sufficient to prove that for any parabolic limit set  $\partial Q \subset \partial A$  and any  $\delta, r > 0$  with  $\delta < \delta_0(r)$ , there exists a constant  $C = C(p, \partial Q, r)$  such that for every  $k \geq 1$

$$\text{Mod}_p^A(\mathcal{F}_{\delta, r}^A(\partial Q), G_k^A) \leq C \cdot \text{Mod}_p(\mathcal{F}_0, G_k).$$

To this end, fix  $\eta$  a non-constant curve in  $\partial Q$  and  $\epsilon > 0$  such that the hypotheses of Theorem 6.12 in  $\partial A$  and of Theorem 8.9 are satisfied. Then there exist two constants  $K$  and  $K'$  depending only on  $p, \eta, r$  and  $\epsilon$  such that for every  $k \geq 1$

$$\text{Mod}_p^A(\mathcal{F}_{\delta, r}^A(\partial Q), G_k^A) \leq K \cdot \text{Mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A) \leq K' \cdot \text{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

Finally, as in the beginning the proof of Theorem 6.12 we can assume without loss of generality that for  $\epsilon > 0$  small enough  $\mathcal{U}_\epsilon(\eta) \subset \mathcal{F}_0$ . Again the multiplicative constant resulting from this assumption only depends on  $d_0$ . This assumption and Proposition 2.3 (1) provide the desired inequality.

The equality of the critical exponents is an immediate consequence of the inequalities between the modulus.  $\square$

**Remark 9.2.** *In the case where  $\Sigma$  is a right-angled Fuchsian building of constant thickness, M. Bourdon gave the explicit value of the conformal dimension of  $\partial \Gamma$ .*

**Theorem 9.3** ([Bou97]). *Let  $\Gamma$  be the graph product associated with a pair  $(C_n, \{\mathbb{Z}/q\mathbb{Z}\}_{i=1, \dots, n})$  where  $C_n$  is a cyclic graph of length  $n \geq 5$  and  $q \geq 2$ , then*

$$\text{Confdim}(\partial \Gamma) = 1 + \frac{\log(q-1)}{\text{Arg cosh } \frac{n-2}{2}}.$$

## 10 Dimension 3 and 4 right-angled buildings with boundary satisfying the CLP

In a well chosen case, the symmetries of the Davis chamber, that extend to the boundary of an apartment, provide a strong control of the weighted modulus. This lead to the proof of the main theorem of this article.

Here we still assume that  $\Gamma$  is of constant thickness  $q \geq 3$ . As usual,  $W$  is the Coxeter group, associated with  $\Gamma$ . As before  $\{G_k^A\}_{k \geq 0}$  and  $\{G_k\}_{k \geq 0}$  are the approximations of  $\partial A$  and  $\partial \Gamma$  provided by Fact 8.2.

In this subsection, we assume that  $W$  is the group generated by the reflections about the faces of a compact right-angled polytope  $D \subset \mathbb{H}^d$ .

Now we write  $\text{Ref}(\mathbb{H}^d)$  the group generated by all the hyperbolic reflections in  $\mathbb{H}^d$ . In the following we designate by  $\text{Ref}(D)$  the stabilizer of  $D$  under the action of  $\text{Ref}(\mathbb{H}^d)$ .

Now, with additional assumptions on the regularity of  $D$ , we prove that  $\partial \Gamma$  satisfies the CLP.

**Theorem 10.1.** *Let  $\Gamma$  be a graph product of constant thickness  $q \geq 3$ . Assume that  $W$  is the group generated by the reflections about the faces of a compact right-angled polytope  $D \subset \mathbb{H}^d$ . Moreover, assume that the quotient of  $D$  by  $R_{ef}(D)$  is a simplex in  $\mathbb{H}^d$ . Then  $\partial \Gamma$  satisfies the CLP.*

Now we assume that the hypotheses of the preceding theorem hold and we use the following notation.

**Notation.**

- $T$  is the hyperbolic simplex in  $\mathbb{H}^d$  isometric to  $D/R_{ef}(D)$ .
- $W_T$  is the hyperbolic reflection group generated by the reflections about the codimension 1 faces of  $T$ .

We notice that  $W$  is a finite index subgroup of  $W_T$ . Indeed,  $W$  is a subgroup of  $W_T$  and they both act discretely on  $\mathbb{H}^d$  with finite co-volume. Then  $W_T$  acts by polyhedral isometries on an apartment of  $\Sigma$ . Indeed, a reflection about a face of  $T$  either preserves  $D$ , or is a reflection about a face of  $D$ . In particular, it preserves the tiling of  $\mathbb{H}^d$  by  $D$ .

Thanks to the constant thickness and the results in the preceding section, we only need to study the usual combinatorial modulus in the apartment to prove the theorem.

**Lemma 10.2.** *Let  $p \geq 1$  and let  $A \in \mathcal{A}p_0(\Sigma)$ . Let  $\eta$  be a non-constant curve in  $\partial A$ . Then there exists a constant  $C = C(p, \eta, \epsilon)$  such that for every  $k \geq 1$*

$$\text{mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq C \cdot \text{mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A).$$

Furthermore, when  $p$  belongs to a compact subset of  $[1, +\infty)$  the constant  $C$  may be chosen independent of  $p$ .

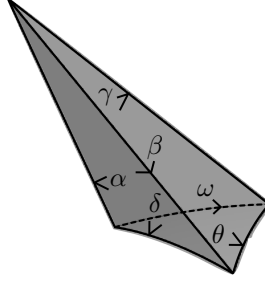


Figure 8: If  $D$  is a dodecahedron,  $T$  is the hyperbolic tetrahedron with dihedral angles  $\alpha = \pi/5, \beta = \pi/3, \gamma = \delta = \omega = \pi/2$  and  $\theta = \pi/4$ ,

*Proof.* To prove this lemma, we first use the fact that  $\partial W_T$  is identified with  $\partial A$  as  $W_T$  acts geometrically on  $A$ . Hence the combinatorial visual metric on  $\partial A$  defines a self-similar metric  $d_{W_T}$  on  $\partial W_T$ . A  $\kappa$ -approximation  $\{G_k^A\}_{k \geq 0}$  of  $\partial A$  induces a  $\kappa$ -approximation on  $\partial W_T$  with same the modulus.

On the other hand,  $\partial W_T$  contains no proper parabolic limit set. Indeed, the Coxeter polytope  $T$  of  $W_T$  is a simplex so all the proper parabolic subgroups of  $W_T$  are finite. In particular, for any non-constant curve  $\eta \subset \partial W_T$ , the smallest parabolic subset containing  $\eta$  is  $\partial W_T$ .

As a consequence, by [BK13, Corollary 6.2.] we get that for every  $\epsilon > 0$ , there exists  $C = C(p, \eta, \epsilon)$  such that for every  $k \geq 1$

$$\text{mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq C \cdot \text{mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A).$$

The Corollary 6.2 in [BK13] is the equivalent for Coxeter groups of our Theorem 6.12 for graph products.  $\square$

*Proof of Theorem 10.1.* We check that the hypotheses of Proposition 3.12 are satisfied. To prove that  $\text{Mod}_1(\mathcal{F}_0, G_k)$  is unbounded, it is enough to prove that there exist  $N$  disjoint curves of diameter larger than  $d_0$  in  $\partial \Gamma$  for every  $N \in \mathbb{N}$  as we did at the beginning of the proof of Theorem 6.13.

Now we let  $p \geq 1$ ,  $\eta$  be a non-constant curve in  $\partial \Gamma$ , and  $\epsilon > 0$ . Without loss of generality, we can assume that there exists  $A \in \mathcal{A}p(\Sigma)$  such that  $\eta \subset \partial A$ . Indeed, as a consequence of Theorem 6.12 if  $\partial P$  is the smallest parabolic limit set containing  $\eta$ . For  $\eta'$  a curve such that  $\partial P$  is the smallest parabolic limit set containing  $\eta'$ , then, up to a multiplicative constant, the behavior of  $\text{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k)$  and  $\text{Mod}_p(\mathcal{U}_\epsilon(\eta'), G_k)$  when  $k$  goes to infinity are the same. The multiplicative constant resulting from this assumption only depends on  $\eta$ .

Using the same arguments as in the beginning of the proof of Theorem 6.12, we can also assume, without loss of generality, that for  $\epsilon > 0$  small enough

- $\mathcal{U}_\epsilon(\eta) \subset \mathcal{F}_0$ ,

- $x_0 \in \text{Ch}(A)$ .

Again the multiplicative constant resulting from these assumptions only depends on  $d_0$ .

Then, thanks to the constant thickness, the inequality of Lemma 10.2 becomes

$$\text{Mod}_p^A(\mathcal{F}_0^A, G_k^A) \leq C \cdot \text{Mod}_p^A(\mathcal{U}_\epsilon^A(\eta), G_k^A),$$

where  $C$  depends only on  $p$ ,  $\eta$  and  $\epsilon$ .

Finally, it is enough to apply Theorem 8.13 to the left-hand term and Theorem 8.9 to the right-hand term of the previous inequality to complete the proof.  $\square$

**Corollary 10.3.** *Let  $\Sigma$  be a building of constant thickness  $q \geq 3$ . Assume that the Coxeter group of  $\Sigma$  is the reflection group of the right-angled dodecahedron in  $\mathbb{H}^3$  or the reflection group of the right-angled 120-cells in  $\mathbb{H}^4$ , then  $\partial\Sigma$  satisfies the CLP.*

**Remark 10.4.** *The hyperbolic 120-cell was described by H.S.M. Coxeter in [Cox73] (see also [Dav08, Appendix B.2.]). It has been used by M.W. Davis to build a compact hyperbolic 4-manifold in [Dav85].*

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